Form factors of boundary fields for A_2-affine Toda field theory

Olalla A. Castro-Alvaredo

Centre for Mathematical Science, City University London,
Northampton Square, London EC1V 0HB, UK

In this paper we carry out the boundary form factor program for the A_2-affine Toda field theory at the self-dual point. The latter is an integrable model consisting of a pair of particles which are conjugated to each other, that is $1 = \bar{2}$, and possessing two bound states resulting from the scattering processes $1 + 1 \to 2$ and $2 + 2 \to 1$. We obtain solutions up to four particle form factors for two families of fields which can be identified with spinless and spin-1 fields of the bulk theory. Previously known as well as new bulk form factor solutions are obtained as a particular limit of ours. Minimal solutions of the boundary form factor equations for all A_n-affine Toda field theories are given, which will serve as starting point for a generalisation of our results to higher rank algebras.

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o.castro-alvaredo@city.ac.uk

1 Introduction

In the context of $1+1$-dimensional integrable quantum field theories (IQFTs), form factors are defined as tensor valued functions representing matrix elements of some local operator $\mathcal{O}(x)$ located at the origin $x=0$ between a multi-particle *in*-state and the vacuum:

$$F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) := \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_n\rangle^{\text{in}}_{\mu_1,\ldots,\mu_n} \quad . \tag{1.1}$$

Here $|0\rangle$ represents the vacuum state and $|\theta_1,\ldots,\theta_n\rangle^{\text{in}}_{\mu_1,\ldots,\mu_n}$ the physical "in" asymptotic states. The latter carry indices μ_i, which are quantum numbers characterizing the various particle species, and depend on the real parameters θ_i, which are called rapidities. The form factors are defined for all rapidities by analytically continuing from some ordering of the latter; a fixed ordering provides a complete basis of states.

The form factor program for integrable models was pioneered by P. Weisz and M. Karowski [1, 2] in the late 70s and thereafter developed to a large extent by F. A. Smirnov, who also formulated some of the consistency equations for form factors and noticed that they corresponded to a particular deformation of the Knizhnik-Zamolodchikov equations [3]. It was found that the form factors of local operators can be obtained as the solutions to a set of consistency equations which characterize the analyticity and monodromy properties of the form factors and constitute what is known as a Riemann-Hilbert problem. It is therefore a priori possible, by solving these equations, to compute all n-particle form factors associated to any local field of a massive IQFT. This is however a very difficult mathematical problem which has only been fully completed for free theories. However, even partial solutions to the problem have proven over the years to be extremely useful for the computation of physical quantities. The reason is that all these quantities are related in a way or another to correlation functions of local fields. At the same time, it is a very well known fact that correlation functions can be expressed as infinite sums depending on the form factors of the fields involved. Although such sums can not be performed analytically, they happen to converge very quickly in terms of the number of particles of the form factors involved. Indeed, convergence is often so good that the knowledge of the two particle form factors is enough to get extremely precise results (see e.g. [4, 5]). This is one of the main reasons why, since it first appeared, the bulk form factor program has attracted so much attention and has been employed successfully for the computation of correlation functions of many integrable models (see [6] for a recent review).

A natural generalisation of these ideas is their extension to integrable theories with boundaries. The study of IQFTs with boundaries goes back to the works of I. V. Cherednik [7] and E. V. Sklyanin [8], where the conditions for a boundary to preserve integrability were established in the form of generalized Yang-Baxter equations which, given the bulk S-matrix, can be solved for the reflection amplitudes off the boundary. These pioneering works were followed by [9, 10, 11], where the authors were concerned with the computation of reflection amplitudes in ATFTs, and by [12], where the crossing property of the reflection amplitudes as well as an explicit realization of the boundary were proposed. The latter work provided in fact the first tool for the computation of form factors in the presence of a boundary. The main idea is to realize the boundary as a "boundary state" located at the origin of time ($t=0$) and expressible in terms of the same creation-annihilation operators used to build the bulk Hilbert-space. The drawback of this approach is that the boundary state is in fact an infinite sum of particle states of the bulk theory, which makes computations very involved, even for free theories [13, 14]. It is worth saying that within this approach, only form factors of fields sufficiently far from the boundary can be obtained, e.g. form factors of fields sitting at the boundary are not accessible. A complementary approach which allows the computation of form factors of fields sitting at the boundary has been proposed recently [15]. It turns out that if one considers the boundary as a point-like object sitting at the origin of space ($x = 0$), then a natural generalisation of the bulk form factor program for IQFTs is possible. We will call this the boundary form factor program. Within this program the form factors of fields sitting at the boundary can be computed for particular models by solving a Riemann-Hilbert problem which is reminiscent of the bulk case. Indeed, as for the bulk case, this Riemann-Hilbert problem is very similar to the set of equations employed previously for the computation of the form factors of integrable spin chains with boundaries [16]-[20]. The latter equations result from a generalisation of the approach developed by M. Jimbo, T. Miwa and their collaborators [21, 22, 23] for bulk models. The first form factors obtained by means of the boundary form factor program were found in [15] for several theories with a single-particle spectrum. An extension of this work followed in [24] where higher particle form factors of the sinh-Gordon model were obtained and in [25] where a precise counting of the number of solutions to the boundary form factor equations of the sinh-Gordon and Yang-Lee models was performed. In this paper we intend to extend this program to a class of multi-particle theories: the A_n-affine Toda field theories.

Affine Toda field theories (ATFTs) have been studied since a long time and have played a prominent role in the development of the field of integrable field theories [26, 27]. Notably, the picture proposed by A. B. Zamolodchikov whereby IQFTs may be regarded as conformal field theories perturbed by relevant fields was first illustrated on the example of the E_8-ATFT [28, 29]. From this example and from extended work on classical Toda theory it was expected that a different theory should exist for each simple Lie algebra and that this Lie algebraic structure would be crucial in the understanding of these models. Subsequently, a lot a work was carried out in order to compute the S-matrices of ATFTs related to each simple Lie algebra. Much of this work was reviewed and extended in [30], where expressions for the S-matrices of all ATFTs can be found. More recently a closed universal formula for the S-matrices of all ATFTs which depends solely on Lie algebraic quantities has been obtained [31].

Amongst the large class of ATFTs the A_n-case is probably the one for which structures are most simple. For example, two-particle scattering amplitudes have poles in the physical sheet whose order is at most two (in contrast to other ATFTs where poles of up to order 12 occur). As a consequence the pole structure of both bulk [32, 33] and boundary form factors is less involved. The S-matrices of the A_n-series were first obtained in [34], where it was suggested that they should be given by the S-matrices of the corresponding minimal A_n-Toda field theories (higher rank generalisations of the Ising model, then known as $\mathbb{Z}_n$-models) obtained originally in [35], multiplied by a CDD factor which contained the dependence on the coupling constant.

In the specific context of ATFTs with boundaries, a great deal of work has been carried out in the last years to compute the corresponding reflection probabilities and to classify the boundary conditions that are consistent with integrability. The first reflection probabilities for ATFTs associated to simply-laced Lie algebras were obtained by A. Fring and R. Köberle in a series of papers [9, 10, 11]. Notably, the last of these papers dealt for the first time with the case of dynamic boundaries. Another series of papers by members of the Durham and York mathematics groups [36, 37, 38] achieved the full classification of boundary conditions which are compatible with integrability. Later on, many authors have computed further reflection amplitudes for various ATFTs (a quite complete list of references can be found in [39]). Given the topic of this paper, it is worth emphasizing those contributions which dealt with the A_n-case. Besides the work cited above, solutions for all A_n-ATFTs in terms of hyperbolic functions were obtained in [36, 40]. The specific A_2-case was studied in detail in [41]. Integral representations of the solutions obtained in [36], as well as solutions for other algebras were given in [42]. Finally, it is worth emphasizing that the A_1-case (that is, the sinh-Gordon model) has, independently from other A_n-ATFTs, attracted a lot of interest as its reflection amplitudes exhibit some properties that single them out from those of all other A_n-ATFTs [43]-[49].

One of the main differences between the solutions in [9, 10, 11] and those in [41, 40, 42, 36] is the fact that whereas the former are invariant under a duality transformation (we will see later what this means) the latter are not. However it was found in [39] that these two types of solutions can be related to each other in a simple fashion (see section 2.2). In fact that work provided the first closed solution for the boundary reflection amplitudes which holds for all ATFTs and which can be related via CDD factors to all known solutions in the literature so far. In this paper we will take the reflection amplitudes computed in [36, 42] as starting point.

The paper is organized as follows: In section 2 we review the expressions and several properties of the S-matrices and reflection amplitudes of the A_n-ATFTs. In section 3 we introduce the boundary form factor program [15] and compute the minimal one- and two-particle form factors for all A_n-ATFTs. We also give a general ansatz for the higher particle form factors, thereby characterizing their pole structure. In section 4 we concentrate on the derivation of recursive equations for the form factors of A_2-ATFT and find solutions for them at the self-dual point, $B = 1$, and up to four particles. We show that some of our solutions correspond to the same class of fields whose bulk form factors were computed by T. Oota in [32]. In addition, we also construct the form factors up to four particles for another class of fields which correspond to spin-1 operators of the bulk theory. In section 5 we present our conclusions and outlook.

2 The A_n-ATFTs : Generalities

The A_n-ATFTs are a family of IQFTs whose exact S-matrices [34], R-matrices [9, 36, 10, 11, 41, 40, 42] and mass spectrum [35] are completely characterized by an underlying Lie algebraic structure related to the simply-laced algebra A_n. The theory has n stable particles in the spectrum with masses

$$m_a = 2m \sin \frac{a\pi}{h} \qquad a = 1, \ldots, n, \tag{2.1}$$

in terms of a fundamental mass scale m and the Coxeter number of A_n, $h = n + 1$. If we arrange these masses into a vector $(m_1, \ldots, m_n)$ then this is the Perron-Frobenius eigenvector of the Cartan matrix of A_n. It is common to relate these particles to the nodes of the Dynkin diagram of A_n. In accordance with the symmetry of the latter, the anti-particle of a particle i corresponds to the $h - i$ node, as shown in the figure:

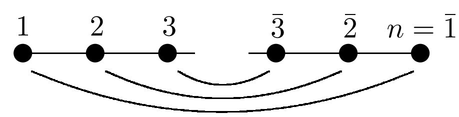

Figure 1: The Dynkin diagram of the simply-laced algebra A_n

2.1 The S-matrices

The S-matrices admit both an integral [50, 51] and a block representation [34]. The integral representation takes the form

$$S_{ab}(\theta, B) = \exp\left[\int_0^\infty \frac{dt}{t}\, \Phi_{ab}(t, B) \sinh\left(\frac{t\theta}{i\pi}\right)\right], \tag{2.2}$$

with

$$\Phi_{ab}(t,B) = \frac{8\sinh\left(\frac{tB}{2h}\right)\sinh\left(\frac{t(2-B)}{2h}\right)\sinh\left(\frac{t\min(a,b)}{h}\right)\sinh\left(\frac{t(h-\max(a,b))}{h}\right)}{\sinh\left(\frac{t}{h}\right)\sinh t}, \tag{2.3}$$

where B is the effective coupling constant which is a function of the coupling constant β which appears in the classical Lagrangian of the theory . By construction the S-matrices are invariant under the transformation $B \to 2-B$, which is equivalent to week-strong duality, $\beta \to 4\pi/\beta$ of the Lagrangian [30]. The point $B=1$ is known as the self-dual point. The kernel above admits an alternative representation, based on the identity

$$\frac{\sinh\left(\frac{t\min(a,b)}{h}\right)\sinh\left(\frac{t(h-\max(a,b))}{h}\right)}{\sinh\left(\frac{t}{h}\right)} = \sum_{\substack{p=|a-b|+1 \\ \text{step } 2}}^{a+b-1} \sinh\left(\frac{t(h-p)}{h}\right), \tag{2.4}$$

which allows us to bring the S-matrix into the block product form

$$S_{ab}(\theta,B) = \prod_{\substack{p=|a-b|+1 \\ \text{step } 2}}^{a+b-1} \{p\}_\theta, \tag{2.5}$$

with blocks

$$\{x\}_\theta = \frac{(x+1)_\theta(x-1)_\theta}{(x+1-B)_\theta(x-1+B)_\theta}, \qquad \text{and} \qquad (x)_\theta = \frac{\sinh\frac{1}{2}\left(\theta+\frac{i\pi x}{h}\right)}{\sinh\frac{1}{2}\left(\theta-\frac{i\pi x}{h}\right)}, \tag{2.6}$$

or

$$\{x\}_\theta = \exp\left[8\int_0^\infty \frac{dt\sinh\left(\frac{tB}{2h}\right)\sinh\left(\frac{t(2-B)}{2h}\right)\sinh t\left(1-\frac{x}{h}\right)}{t\sinh t}\sinh\left(\frac{t\theta}{i\pi}\right)\right]. \tag{2.7}$$

From the block form above it is easy to see that these S-matrices possess simple and double poles. As usual in this context, simple poles are related to the presence of bound states in the theory which result from fusing processes of the form $a+b\to\bar{c}$ whenever $a+b+c$ equals either h or $2h$. If such fusing occurs the amplitude $S_{ab}(\theta)$ will have a pole $\theta = iu_{ab}^{\bar{c}}$ located at

$$u_{ab}^{\bar{c}} = \begin{cases} \frac{\pi(a+b)}{h} & \text{if} \quad a+b+c=h \\ \frac{\pi(2h-a-b)}{h} & \text{if} \quad a+b+c=2h \end{cases}. \tag{2.8}$$

Closed formulae for the associated S-matrix residua, which are closely related to the on-shell three-point coupling $\Gamma_{ab}^{\bar{c}}$, can be easily obtained. For example, for the process $1+1\to 2$:

$$(\Gamma_{11}^2)^2 = -i\lim_{\theta\to iu_{11}^2}(\theta-iu_{11}^2)S_{11}(\theta) = \frac{2\sin\left(\frac{2\pi}{h}\right)\sin\left(\frac{(2-B)\pi}{2h}\right)\sin\left(\frac{B\pi}{2h}\right)}{\sin\left(\frac{(2+B)\pi}{2h}\right)\sin\left(\frac{(4-B)\pi}{2h}\right)}. \tag{2.9}$$

2.2 The reflection amplitudes

A particular set of solutions for the R-matrices of A_n-ATFTs was constructed in [36] in a block product representation,

$$R_a(\theta,B)=R_{\bar{a}}(\theta,B)=\prod_{p=1}^{a}\|p\|_\theta, \tag{2.10}$$

with

$$\|x\|_\theta=\frac{\left(\frac{x-1}{2}\right)_\theta\left(\frac{x+1}{2}-h\right)_\theta}{\left(\frac{x-1+B}{2}-h\right)_\theta\left(\frac{x+1-B}{2}\right)_\theta}. \tag{2.11}$$

The corresponding integral representation was given in [42]

$$R_a(\theta,B)=R_{\bar{a}}(\theta,B)=\exp\left[\int_0^\infty\frac{dt}{t}\rho_a(t,B)\sinh\left(\frac{t\theta}{i\pi}\right)\right], \tag{2.12}$$

with

$$\begin{aligned}\rho_a(t,B)=\rho_{\bar{a}}(t,B) &= \frac{8\sinh\left(\frac{t(2-B)}{4h}\right)\sinh\left(\frac{t(2h+B)}{4h}\right)\sinh\left(\frac{ta}{2h}\right)\sinh\left(\frac{t(h-a)}{2h}\right)}{\sinh\left(\frac{t}{2h}\right)\sinh t}\\ &= \frac{8\sinh\left(\frac{t(2-B)}{4h}\right)\sinh\left(\frac{t(2h+B)}{4h}\right)}{\sinh t}\sum_{p=1}^{a}\sinh\left(\frac{t(h-2p+1)}{2h}\right).\end{aligned} \tag{2.13}$$

A feature of these solutions is that, contrary to the S-matrices (2.2), they are not invariant under the duality transformation $B\to 2-B$. As a result, they have the property of having different classical limits in the weak ($B\to 0$) and strong ($B\to 2$) coupling limits, namely

$$R_a(\theta,0)=\prod_{b=1}^{n}S_{ab}(\theta,1),\qquad R_a(\theta,2)=1. \tag{2.14}$$

This means that in the classical limit the amplitudes $R_i(\theta,0)$ correspond to a theory with fixed Dirichlet boundary conditions whereas $R_i(\theta,2)$ correspond to a theory with von Neumann boundary conditions [36, 37, 38]. How the breaking of duality invariance occurs can be explicitly seen by writing the corresponding R-matrices as

$$R_a(\theta,B)=\tilde{R}_a(\theta,B)\prod_{b=1}^{n}S_{ba}\left(\theta,1-\frac{B}{2}\right), \tag{2.15}$$

where $\tilde{R}$ are the duality invariant R-matrices constructed in [39] and the identity (2.15) was also found there.*

*The R-matrices constructed in [39] admit an integral representation of the form (2.12) with

$$\rho_j(t,B)=\frac{8\sinh\left(\frac{t(2-B)}{4h}\right)\sinh\left(\frac{tB}{4h}\right)\sinh\left(\frac{t(1-h)}{2h}\right)\sinh\left(\frac{tj}{2h}\right)\sinh\left(\frac{t(h-j)}{2h}\right)}{\sinh^2\left(\frac{t}{2h}\right)\sinh t}$$

Notice that this kernel differs from the one given in [39] which contained several typos.

3 The boundary form factor consistency equations

A set of consistency equations whose solutions are the form factors of boundary fields of an IQFT was recently proposed in [15]. There the equations were presented for the special case of a theory with a single particle species, but it is trivial to write them for the general case. The first three relations

$$
\begin{aligned}
F_n^{\mathcal{O}|\ldots\mu_i\mu_{i+1}\ldots}(\ldots,\theta_i,\theta_{i+1},\ldots) &= S_{\mu_i\mu_{i+1}}(\theta_i-\theta_{i+1})F_n^{\mathcal{O}|\ldots\mu_{i+1}\mu_i\ldots}(\ldots,\theta_{i+1},\theta_i,\ldots), &(3.1)\\
F_n^{\mathcal{O}|\mu_1\ldots\mu_{n-1}\mu_n}(\theta_1,\ldots,\theta_{n-1},\theta_n) &= R_{\mu_n}(\theta_n)F_n^{\mathcal{O}|\mu_1\ldots\mu_{n-1}\mu_n}(\theta_1,\ldots,\theta_{n-1},-\theta_n), &(3.2)\\
F_n^{\mathcal{O}|\mu_1\mu_2\ldots\mu_n}(\theta_1,\theta_2\ldots,\theta_n) &= R_{\mu_1}(i\pi-\theta_1)F_n^{\mathcal{O}|\mu_1\mu_2\ldots\mu_n}(2\pi i-\theta_1,\theta_2,\ldots,\theta_n), &(3.3)
\end{aligned}
$$

express the fact that exchanging two particles in a form factor amounts to the scattering of those particles, which is appropriately taken into account by the two-particle S-matrix. In addition if one of the particles a form factor depends upon is scattered off the boundary with probability $R_{\mu_i}(\theta_i)$, the value of the form factor should remain unchanged, which is expressed by the second equation. Finally, the particle might be scattered off the boundary after a $2\pi i$ rotation, which is expressed by the last equation. For $n=1$ and $n=2$ the minimal solutions to these equations (that is, solutions with no poles in the physical sheet) will play a crucial role in the formulation of a general ansatz for the n-particles form factors and details on their computation will be provided in the coming sections.

In addition to these three equations we have two more relations which fix the pole structure of the form factors: the kinematic residue equation

$$
\begin{aligned}
\lim_{\bar{\theta}_0\to\theta_0} F_{n+2}^{\mathcal{O}|\bar{\mu}\mu\mu_1\ldots\mu_n}(\bar{\theta}_0+i\pi,\theta_0,\theta_1,\ldots,\theta_n) &= i\left(1-\prod_{k=1}^{n}S_{\mu\mu_k}(\theta_0-\theta_k)S_{\mu\mu_k}(\theta_0+\theta_k)\right)\\
&\times\ F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n), \qquad (3.4)
\end{aligned}
$$

and the bound state residue equation

$$
\lim_{\epsilon\to 0}\epsilon F_{n+2}^{\mathcal{O}|ab\mu_1\ldots\mu_n}(\theta+\frac{iu_{ab}^c}{2}+\epsilon,\theta-\frac{iu_{ab}^c}{2}-\epsilon,\theta_1,\ldots,\theta_n) = i\Gamma_{ab}^c F_{n+1}^{\mathcal{O}|c\mu_1\ldots\mu_n}(\theta,\theta_1,\ldots,\theta_n), \quad (3.5)
$$

where particles a and b fuse to produce particle c which implies that the scattering amplitude $S_{ab}(\theta)$ has a pole at $\theta=iu_{ab}^c$ with residue given by $(\Gamma_{ab}^c)^2$.

3.1 The minimal one-particle form factors

As explained in [15], starting with the boundary reflection amplitudes, it is possible to find a minimal solution, $w_a(\theta)$, to the one-particle form factor equations[†]

$$
w_a(\theta)=R_a(\theta)w_a(-\theta) \qquad \text{and} \qquad w_a(\theta+i\pi)=R_a(-\theta)w_a(i\pi-\theta), \quad (3.6)
$$

which takes the form

$$
w_a(\theta)=g_a(\theta)g_a(i\pi-\theta), \quad (3.7)
$$

where $g_a(\theta)$ satisfies the equations

$$
g_a(\theta)=R_a(\theta)g_a(-\theta) \qquad \text{and} \qquad g_a(\theta)=g_a(2\pi i-\theta). \quad (3.8)
$$

[†]From here onwards we will omit the explicit dependence of the reflection amplitudes, S-matrix and form factors in B.

These are the same equations as for the two-particle minimal form factor, with the S-matrix replaced by the reflection amplitudes. Therefore they can be solved in a systematic way, namely

$$g_a(\theta) = g_{\bar{a}}(\theta, B) = \exp\left[-\frac{1}{2}\int_0^\infty \frac{dt}{t\sinh t}\rho_a(t,B)\cosh t\left(1+\frac{i\theta}{\pi}\right)\right]. \tag{3.9}$$

We can write (3.9) as an infinite product of Euler's gamma functions by employing the identities:

$$\int_0^\infty \frac{dt}{t}\frac{\sinh(\alpha t)\sinh(\beta t)e^{-\gamma t}}{\sinh(ut)} = \frac{1}{2}\log\left[\frac{\Gamma\left(\frac{\alpha+\beta+\gamma+u}{2u}\right)\Gamma\left(\frac{-\alpha-\beta+\gamma+u}{2u}\right)}{\Gamma\left(\frac{-\alpha+\beta+\gamma+u}{2u}\right)\Gamma\left(\frac{\alpha-\beta+\gamma+u}{2u}\right)}\right], \tag{3.10}$$

and

$$\frac{1}{\sinh t} = 2e^{-t}\sum_{k=0}^\infty e^{-2kt}. \tag{3.11}$$

We obtain

$$g_a(\theta,B) = g_{\bar{a}}(\theta,B) = \prod_{p=1}^a \frac{\sinh\frac{1}{2}\left(\theta - \frac{i\pi(2p-B)}{2h}\right)\cosh\frac{1}{2}\left(\theta-\frac{i\pi(B+2p-2)}{2h}\right)\varrho^1_{p-1}(\theta)\varrho^{\frac{1}{2}}_p(\theta)}{\sinh\frac{1}{2}\left(\theta-\frac{i\pi(p-1)}{h}\right)\cosh\frac{1}{2}\left(\theta-\frac{i\pi p}{h}\right)\varrho^1_{\frac{2p-B}{2}}(\theta)\varrho^{\frac{1}{2}}_{\frac{B+2p-2}{2}}(\theta)}, \tag{3.12}$$

where

$$\varrho^n_x(\theta) = \prod_{k=1}^\infty \frac{\Gamma\left(k+n-\frac{x}{2h}+\frac{i\theta}{2\pi}\right)\Gamma\left(k+n-\frac{x}{2h}-\frac{i\theta}{2\pi}\right)}{\Gamma\left(k-n+\frac{x}{2h}-\frac{i\theta}{2\pi}\right)\Gamma\left(k-n+\frac{x}{2h}+\frac{i\theta}{2\pi}\right)}, \qquad x\in[0,2h), \quad n\in\mathbb{Q}. \tag{3.13}$$

The following identity will be useful for later purposes:

$$w_{\bar{a}}(\theta+i\pi)w_a(\theta) = \prod_{p=1}^a \frac{\varrho^0_{2p}(2\theta)\varrho^1_{2p-2}(2\theta)}{\varrho^0_{B+2p-2}(2\theta)\varrho^1_{2p-B}(2\theta)}. \tag{3.14}$$

3.2 The two-particle form factors

The two-particle form factors are the solutions of the two-particle form factor equations

$$F_2^{ab}(\theta_1,\theta_2) = S_{ab}(\theta)F_2^{ba}(\theta_2,\theta_1) = R_b(\theta_2)F_2^{ab}(\theta_1,-\theta_2) = R_a(i\pi-\theta_1)F_2^{ab}(2\pi i-\theta_1,\theta_2). \tag{3.15}$$

It was shown in [15] that solutions to these equations take the general form

$$F_2^{ab}(\theta_1,\theta_2) = f_{ab}(\theta_{12})f_{ab}(\hat{\theta}_{12})w_a(\theta_1)w_b(\theta_2)\Phi(y_1,y_2), \quad \text{with} \quad y_i = e^{\theta_i}+e^{-\theta_i}. \tag{3.16}$$

Here $\theta_{12} = \theta_1 - \theta_2$, $\hat{\theta}_{12} = \theta_1 + \theta_2$ and $f_{ab}(\theta)$ is the bulk two-particle minimal form factor, that is a solution of the equations

$$f_{ab}(\theta) = S_{ab}(\theta)f_{ba}(-\theta) = f_{ab}(2\pi i - \theta), \tag{3.17}$$

with no poles in the physical strip $\mathrm{Im}(\theta) \in [0,\pi]$. The function $\Phi(y_1,y_2)$ characterizes the particular operator and is a symmetric function of the variables y_1, y_2, which is the constraint given in [15] for models with a single particle spectrum. In particular, the minimal boundary two-particle form factors correspond to $\Phi(y_1,y_2) = 1$. Solutions to (3.17) for A_n-ATFTs were first found by T. Oota [32] in the form of an infinite product of Γ-functions. Such representation can be obtained in the usual way: for an S-matrix which admits an integral representation of the type (2.2) it is easy to show that a solution to (3.17) is given by

$$f_{ab}(\theta) = \exp\left[-\frac{1}{2}\int_0^\infty \frac{dt}{t\sinh t}\,\Phi_{ab}(t,B)\cosh t\left(1+\frac{i\theta}{\pi}\right)\right]. \tag{3.18}$$

As for the one-particle form factors, we can use (3.10) and (3.11) to express the form factor above as an infinite product of gamma functions

$$f_{ab}(\theta) = \prod_{\substack{p\,=\,|a-b|+1\\ \text{step } 2}}^{a+b-1} \frac{\varrho^1_{p+1}(\theta)\varrho^1_{p-1}(\theta)}{\langle -p\rangle_\theta\,\varrho^1_{p+1-B}(\theta)\varrho^1_{p-1+B}(\theta)}, \tag{3.19}$$

where

$$\langle p\rangle_\theta = \frac{\sinh\frac{1}{2}\left(\theta+\frac{i\pi(p-1)}{h}\right)\sinh\frac{1}{2}\left(\theta+\frac{i\pi(p+1)}{h}\right)}{\sinh\frac{1}{2}\left(\theta+\frac{i\pi(p-1+B)}{h}\right)\sinh\frac{1}{2}\left(\theta+\frac{i\pi(p+1-B)}{h}\right)}. \tag{3.20}$$

It is easy to check that the function inside the product above equals the function $F_p^{\min}(\theta)$ introduced by T. Oota in [32], as it should be. A useful identity involving the minimal form factors above is

$$f_{ab}(\theta)f_{\bar{a}b}(\theta+i\pi) = \prod_{\substack{p\,=\,|a-b|+1\\ \text{step } 2}}^{a+b-1} \langle p\rangle_\theta\,, \tag{3.21}$$

which was also obtained in [32]. In addition, it is easy to show from (3.19) that

$$f_{\bar{a}a}(2\theta+i\pi) = \prod_{p=1}^{\min(a,\bar{a})} \frac{\varrho^0_{B+2p-2}(2\theta)\varrho^0_{2p-B}(2\theta)}{\varrho^0_{2p}(2\theta)\varrho^0_{2p-2}(2\theta)}. \tag{3.22}$$

Combining this with (3.14) we find that

$$f_{\bar{a}a}(2\theta+i\pi)w_{\bar{a}}(\theta+i\pi)w_a(\theta) = \prod_{p=1}^{\min(\bar{a},a)} \frac{\sinh\left(\theta+\frac{i\pi(p-1)}{h}\right)\sinh\left(\theta-\frac{i\pi(p-1)}{h}\right)}{\sinh\left(\theta+\frac{i\pi(B-2p)}{2h}\right)\sinh\left(\theta-\frac{i\pi(B-2p)}{2h}\right)}. \tag{3.23}$$

Also

$$\frac{f_{11}(\theta+\frac{i\pi}{h})f_{11}(\theta-\frac{i\pi}{h})}{f_{21}(\theta)} = \frac{\sinh\frac{1}{2}\left(\theta+\frac{i\pi}{h}\right)\sinh\frac{1}{2}\left(\theta-\frac{i\pi}{h}\right)}{\sinh\frac{1}{2}\left(\theta+\frac{i\pi(1-B)}{h}\right)\sinh\frac{1}{2}\left(\theta-\frac{i\pi(1-B)}{h}\right)}. \tag{3.24}$$

These identities will be very important in order to bring the kinematic and bound state residue equations into a simple form.

3.3 Higher particle form factors

For reflection matrices which have no poles at $\theta = i\pi/2$ (as is our case here) it is natural to make the following ansatz for the n-particle form factors

$$F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n) = H_n^{\mathcal{O}|\mu_1\ldots\mu_n} Q_n^{\mathcal{O}|\mu_1\ldots\mu_n}(y_1,\ldots,y_n)\prod_{k=1}^{n} w_{\mu_k}(\theta_k) \times \prod_{1\leq i<j\leq n} \frac{f_{\mu_i\mu_j}(\theta_{ij}) f_{\mu_i\mu_j}(\hat{\theta}_{ij})}{P_{\mu_i\mu_j}(\theta_{ij}) P_{\mu_i\mu_j}(\hat{\theta}_{ij})(y_i+y_j)^{\delta_{\mu_i\bar{\mu}_j}}}, \qquad (3.25)$$

where $H_n^{\mathcal{O}|\mu_1\ldots\mu_n}$ are constants which are invariant under any permutation of the indices $\mu_1,\ldots,\mu_n$, and $Q_n^{\mathcal{O}|\mu_1\cdots\mu_n}(y_1,\ldots,y_n)$ are entire functions of $y_i = 2\cosh\theta_i$ with $i = 1,\ldots,n$. The factors in the denominator encode the full pole structure of the form factors. More precisely, the kinematic poles are encoded in the $(y_i + y_j)$ factor, whereas the functions $P_{ab}(\theta)$ account for the bound state poles in the theory. The form of these functions was determined in [32] for the bulk case. They are exactly the same in the boundary theory and can be written as,

$$P_{ab}(\theta) = P_{\bar{a}\bar{b}}(\theta) = \prod_{\substack{p=|a-b|+1,\, p\neq h-1 \\ \text{step } 2}}^{a+b-1} 2\left[\cosh\theta - \cos\left(\frac{\pi(p+1)}{h}\right)\right], \qquad (3.26)$$

which is equivalent to Oota's expression (up to constant factors). The value $p = h-1$ needs to be excluded as it would produce extra poles at $\theta = i\pi$. The main difference in the way of encoding the bound state pole structure with respect to the bulk theory is the presence of products $P_{ab}(\theta_{ab})P_{ab}(\hat{\theta}_{ab})$ rather than a single factor $P_{ab}(\theta_{ab})$. This must be so in order to guarantee invariance under change of sign of one of the rapidities. Such prescription was already used in [15] for the Yang-Lee theory for which (3.26) is simply $2\cosh\theta + 1$ (corresponding to $a = b = 1$ and $h = 3$) and the same holds for the A_2-ATFT.

The ansatz (3.25) satisfies all form factor consistency equations by construction, provided that $Q_n(y_1,\ldots,y_n)$ has the extra property

$$Q_n^{\mathcal{O}|\cdots\mu_a\mu_{a+1}\cdots}(\ldots,y_a,y_{a+1},\ldots) = Q_n^{\mathcal{O}|\cdots\mu_{a+1}\mu_a\cdots}(\ldots,y_{a+1},y_a,\ldots). \qquad (3.27)$$

From this property we deduce that $Q_n^{\mathcal{O}|\mu_1\cdots\mu_n}(y_1,\ldots,y_n)$ must be a symmetric function of all variables related to the same particle type. Therefore we expect that the corresponding form factors can be expressed in terms of elementary symmetric polynomials on the variables related to each particle type. Plugging the ansatz (3.25) into the kinematic residue equation we obtain recursive relations for the polynomials $Q_n^{\mathcal{O}|\mu_1\cdots\mu_n}(y_1,\ldots,y_n)$ of the form

$$Q_{n+2}^{\mathcal{O}|\bar{\mu}\mu\mu_1\cdots\mu_n}(-y,y,y_1,\ldots,y_n) = P_n(y,y_1,\ldots,y_n) Q_n^{\mathcal{O}|\mu_1\cdots\mu_n}(y_1,\ldots,y_n), \qquad (3.28)$$

where $y = 2\cosh\theta$ and $P_n(y,y_1,\ldots,y_n)$ is a polynomial on the variables $y,y_1,\ldots,y_n$. In addition, if two particles a,b fuse to produce a particle c, then also a second set of equations of the form

$$Q_{n+2}^{\mathcal{O}|ab\mu_1\ldots\mu_n}(y_+,y_-,y_1,\ldots,y_n) = W_n(y,y_1,\ldots,y_n) Q_{n+1}^{\mathcal{O}|c\mu_1\cdots\mu_n}(y,y_1,\ldots,y_n), \qquad (3.29)$$

must be satisfied, where $y_\pm = 2\cosh\left(\theta \pm \frac{iu_{ab}^c}{2}\right)$ and $W_{n+1}(y,y_1,\ldots,y_n)$ is a polynomial of its variables. We will now proceed to compute the polynomials P_n and W_n for the simplest model of this class: the A_2-ATFT.

4 Boundary form factors for A_2-ATFT

We will now write down the equations (3.28)-(3.29) for the particular case $h=3$ and $n=2$. We have then a two-particle theory with $\bar{1}=2$ and the following fusing processes

$$1+1 \to 2 \quad \text{with} \quad u_{11}^2 = \frac{2\pi}{3}, \tag{4.1}$$

$$2+2 \to 1 \quad \text{with} \quad u_{22}^1 = \frac{2\pi}{3}. \tag{4.2}$$

Before writing down the recursive equations for this case, it is convenient to introduce some notation. We will write $F_{m+n}^{\mathcal{O}|m,n}(\{y\}_m;\{y'\}_n)$ to denote an $n+m$-particle form factor where m particles are of type 1 and n particles are of type 2. We choose to order indices in such a way that all particles of type 1 appear first and all particles of type 2 appear at the end. In addition, we will group all variables related to particles of type 1 or particles of type 2 into sets $\{y\}_m = \{y_1,\ldots,y_m\}$ and $\{y'\}_n = \{y'_1,\ldots,y'_n\} = \{y_{m+1},\ldots,y_{n+m}\}$, respectively. Similar notation will be used for the polynomials Q. Employing this notation, we can rewrite equation (3.28) as follows

$$Q_{m+n+2}^{\mathcal{O}|m+1,n+1}(y,\{y\}_m;-y,\{y'\}_n) = P_{m+n}(y,\{y\}_m,\{y'\}_n)Q_{m+n}^{\mathcal{O}|m,n}(\{y\}_m;\{y'\}_n), \tag{4.3}$$

where

$$P_{n+m}(y,\{y\}_m,\{y'\}_n) = a(y)\sum_{k,p,r=0}^{m}\sum_{a,b,c=0}^{n}\left(\frac{\omega_-^{m-k}\eta_+^{m-p}\lambda_+^{m-r}}{\tilde{\omega}_-^{a-n}\tilde{\eta}_+^{b-n}\tilde{\lambda}_+^{c-n}} - \frac{\omega_+^{m-k}\eta_-^{m-p}\lambda_-^{m-r}}{\tilde{\omega}_+^{a-n}\tilde{\eta}_-^{b-n}\tilde{\lambda}_-^{c-n}}\right)\sigma_r\sigma_p\sigma_k\hat{\sigma}_a\hat{\sigma}_b\hat{\sigma}_c, \tag{4.4}$$

with

$$\begin{aligned}
a(y) &= \frac{\cosh(2\theta)-\cos\left(\frac{\pi(2-B)}{3}\right)}{i\sqrt{3}\sinh\theta}, && (4.5)\\
\omega_\pm &= -2\cosh\left(\theta \pm \frac{2\pi i}{3}\right), \quad \tilde{\omega}_\pm = -2\cosh\left(\theta\pm\frac{\pi i}{3}\right), && (4.6)\\
\eta_\pm &= -2\cosh\left(\theta \pm \frac{i\pi(2-B)}{3}\right), \quad \lambda_\pm = -2\cosh\left(\theta\pm\frac{i\pi B}{3}\right), && (4.7)\\
\tilde{\eta}_\pm &= -2\cosh\left(\theta \pm \frac{i\pi(3-B)}{3}\right), \quad \tilde{\lambda}_\pm = -2\cosh\left(\theta\pm\frac{i\pi(1+B)}{3}\right), && (4.8)
\end{aligned}$$

and σ and $\hat{\sigma}$ are elementary symmetric polynomials on the variables $\{y\}_m$ and $\{y'\}_n$ respectively. These polynomials can be defined by means of the generating function,

$$\prod_{k=1}^{n}(x+y_i) = \sum_{k=0}^{n} x^{n-k}\sigma_k, \tag{4.9}$$

where the subscript k is the degree of the polynomial σ_k which depends on the n variables $y_1,\ldots,y_n$. Also from the kinematic residue equations it follows that

$$H_{m+n+2}^{\mathcal{O}|m+1,n+1} = \sqrt{3}f_{21}(i\pi)^{-1}H_{m+n}^{\mathcal{O}|m,n}. \tag{4.10}$$

In addition, due to the presence of bound states there is an additional set of recursive equations of the form (3.29). Employing our new notation, we will write these equations as

$$Q_{m+n+2}^{\mathcal{O}|m+2,n}(y_+,y_-,\{y\}_m;\{y'\}_n) = W_{m+n}(y,\{y\}_m,\{y'\}_n)Q_{m+n+1}^{\mathcal{O}|m,n+1}(\{y\}_m;y,\{y'\}_n), \tag{4.11}$$

where

$$W_{m+n}(y,\{y\}_m,\{y'\}_n)=g(B)\left(y^2-2-2\cos\left(\frac{\pi B}{3}\right)\right)\sum_{k,r,s=0}^{m}y^{m-k}\tau_+^{m-r}\tau_-^{m-s}\sigma_k\sigma_r\sigma_s, \quad (4.12)$$

with

$$\tau_\pm=-2\cosh\left(\theta\pm\frac{i\pi(1-B)}{3}\right), \qquad g(B)=\frac{2\Gamma_{11}^2}{3^{1/4}}\sin\left(\frac{\pi(2+B)}{6}\right). \quad (4.13)$$

Notice that $g(1)=1$ and that W_{m+n} does only depend on the variables related to the type 1 particles. This surprising property is due to (3.24) and to the following identities

$$\frac{f_{11}(2\theta)w_1(\theta+\frac{i\pi}{3})w_1(\theta-\frac{i\pi}{3})}{w_1(\theta)P_{11}(2\theta)}=\frac{1}{y^2-2-2\cos\left(\frac{B\pi}{3}\right)}, \quad (4.14)$$

$$\frac{f_{12}(\theta+\frac{i\pi}{3})f_{12}(\theta-\frac{i\pi}{3})}{f_{22}(\theta)}=1, \qquad P_{12}(\theta)=1, \quad (4.15)$$

which hold specifically for the $h=3$ case. The bound state residue equation provides also the following recursive relations between the constants:

$$H_{n+m+2}^{\mathcal{O}|m+2,n}=-\sqrt{\frac{\sqrt{3}}{f_{21}(i\pi)}}H_{m+n+1}^{\mathcal{O}|m,n+1}, \quad (4.16)$$

were we have used the identity

$$f_{21}(i\pi)=\frac{4f_{11}(2\pi i/3)^2}{3}\sin^2\left(\frac{\pi(2+B)}{6}\right), \quad (4.17)$$

which is valid for $h=3$. Equations (4.10)-(4.16) are solved by

$$H_{2n+3k}^{\mathcal{O}|n,n+3k} = (-1)^k\left[\frac{\sqrt{3}}{f_{21}(i\pi)}\right]^{n+3k/2}H_0^{\mathcal{O}|0,0}, \quad (4.18)$$

$$H_{2n+3k+1}^{\mathcal{O}|n,n+1+3k} = (-1)^k\left[\frac{\sqrt{3}}{f_{21}(i\pi)}\right]^{n+3k/2}H_0^{\mathcal{O}|0,1}, \quad (4.19)$$

$$H_{2n+3k+2}^{\mathcal{O}|n,n+2+3k} = (-1)^{k+1}\left[\frac{\sqrt{3}}{f_{21}(i\pi)}\right]^{n+(3k+1)/2}H_0^{\mathcal{O}|0,1}, \quad (4.20)$$

for $k\in\mathbb{Z}^+\cup\{0\}$.

4.1 Solving the form factor recursive equations

Finding solutions to (4.3)-(4.11) is very involved, even for the lower particle form factors. For this reason here we have made a further simplification and consider only the $B=1$ case. We will start by introducing some useful definitions. We define the order of the form factor $F_{m+n}^{\mathcal{O}|m,n}$ as the number $[F_{m+n}^{\mathcal{O}|m,n}]$ which characterizes the asymptotic behaviour

$$\lim_{s\to\infty}F_{m+n}^{\mathcal{O}|m,n}(\{\theta+s\}_m;\{\theta+s\}_n)\sim e^{s[F_{m+n}^{\mathcal{O}|m,n}]}, \quad (4.21)$$

where by $\theta + s$ we mean that each rapidity is shifted by the same amount s. If we were dealing with bulk form factors, this would be simply the spin of the operator under investigation. The order of the polynomial $Q_{m+n}^{\mathcal{O}|m,n}$ can be defined in a similar fashion.

It will also be useful to introduce some particular combinations of elementary symmetric polynomials, which we will denote by $K^{[m,0]}$ and $Z^{[m,n]}$ and which are defined by the following properties:

$$Z^{[m+1,n+1]}(y, \{y\}_m; -y, \{y'\}_n) = 0, \qquad K^{[m+2,0]}(y_+, y_-, \{y\}_m) = 0, \tag{4.22}$$

namely, they are the kernels of the equations (4.3)-(4.11) for $B = 1$. As it turns out, the $K^{[1,n]}$ and $K^{[n,1]}$ polynomials admit a simple closed form

$$Z^{[1,n]} = \sum_{k=0}^{n} \hat{\sigma}_k \sigma_1^{n-k}, \qquad Z^{[n,1]} = \sum_{k=0}^{n} \sigma_k \hat{\sigma}_1^{n-k}, \tag{4.23}$$

and each of the terms in the sums has order n. Here we will also need

$$Z^{[2,2]} = \hat{\sigma}_2^2 + \hat{\sigma}_1\hat{\sigma}_2\sigma_1 + \hat{\sigma}_2\sigma_1^2 + \hat{\sigma}_1^2\sigma_2 - 2\hat{\sigma}_2\sigma_2 + \hat{\sigma}_1\sigma_1\sigma_2 + \sigma_2^2. \tag{4.24}$$

These polynomials are in fact the same introduced in [32] and denoted there by $K^{[m,n]}$. The K-polynomials defined by the property (4.22) become soon rather complex and no obvious pattern in terms of elementary symmetric polynomials seems to emerge. For example:

$$\begin{aligned}
K^{[2,0]} &= 3 - \sigma_1^2 + \sigma_2, && (4.25)\\
K^{[3,0]} &= \sigma_1^3(3\sigma_1 + \sigma_3) - \sigma_1^2(3 + \sigma_2)(6 + \sigma_2) + (3 + \sigma_2)^3, && (4.26)\\
K^{[4,0]} &= -3K^{[2,0]}(3\sigma_1^2 - (3 + \sigma_2)^2)^2 + K^{[3,0]}\sigma_3(9\sigma_1 + \sigma_3) \\
&\quad -3\sigma_3((\sigma_1((3 + \sigma_2)^2 - 3\sigma_1^2) - 4\sigma_3(3 + \sigma_2))(3 + \sigma_2) + (\sigma_1^4 + 4(3 + \sigma_2)^2)\sigma_3) \\
&\quad +\sigma_4(-27\sigma_1^4 + (3 - \sigma_2)(3 + \sigma_2)^3 + \sigma_1^2(3 + \sigma_2)(27 + \sigma_2(6 + \sigma_2)) && (4.27)\\
&\quad -\sigma_1\sigma_3(-9 + \sigma_2^2 + 3\sigma_1(6\sigma_1 + \sigma_3)) + ((3 + \sigma_2)(3 + 2\sigma_2) + 3\sigma_1(3\sigma_1 + \sigma_3) - \sigma_4)\sigma_4).
\end{aligned}$$

Similarly, we define the $K^{[0,n]}$ polynomials as identical to the ones above up to the replacement $\sigma \to \hat{\sigma}$. Like those, they have order $n(n-1)$. In this case, not all terms involved are of order $n(n-1)$ but only the leading ones. In fact if we select out only the leading terms we get a new set of polynomials which will be related to structures occurring in the bulk form factors:

$$\begin{aligned}
K_{\text{bulk}}^{[2,0]} &= \sigma_2 - \sigma_1^2, && (4.28)\\
K_{\text{bulk}}^{[3,0]} &= \sigma_1^3\sigma_3 + \sigma_2^2 K_{\text{bulk}}^{[2,0]}, && (4.29)\\
K_{\text{bulk}}^{[4,0]} &= \sigma_3^2 K_{\text{bulk}}^{[3,0]} - \sigma_2^3\sigma_4 K_{\text{bulk}}^{[2,0]} - \sigma_4\sigma_1\sigma_3(\sigma_2^2 + 3\sigma_1\sigma_3 - 3\sigma_4) + \sigma_4^2(2\sigma_2^2 - \sigma_4). && (4.30)
\end{aligned}$$

These "bulk" polynomials are in fact equivalent (up to a sign) to the polynomials $B_{1[m,n]}$ introduced in [32] and can be expressed as determinants of elementary symmetric polynomials in the way described there. We have now all the tools to find explicit solutions to the form factor equations.

4.2 Form factors of "spinless" fields

In this subsection we will compute all form factors up to four particles of a family of fields, which we will denote by $\mathcal{O}_2$ and which we will try to identify later. In the title of this subsection we have used the word "spinless" to indicate that the order of the form factors associated to these fields is zero and therefore they should correspond to spinless fields in the bulk theory. Indeed, it will turn out that this family of operators is nothing but the boundary counterpart of the fields whose form factors were obtained by T. Oota [32]. The condition $[F_{m+n}^{\mathcal{O}_2|m,n}] = 0$ fixes the order of each polynomial $Q_{m+n}^{\mathcal{O}|m,n}$ as

$$[Q_{m+n}^{\mathcal{O}|m,n}] = m^2 + (m+n)(n-1). \tag{4.31}$$

4.2.1 Vacuum expectation values and 1-particle form factors

Here we will normalize $Q_0^{\mathcal{O}_2|0,0} = 1$ and choose

$$Q_1^{\mathcal{O}_2|0,1} = A_{[0,1]}, \tag{4.32}$$

with $A_{[0,1]}$ an arbitrary constant.

4.2.2 Two-particle form factors

The recursive equations for the two-particle form factors are:

$$\begin{aligned} Q_2^{\mathcal{O}_2|1,1}(-y;y) &= 0, && (4.33)\\ Q_2^{\mathcal{O}_2|2,0}(y_+,y_-) &= (y^2-3)A_{[0,1]}. && (4.34) \end{aligned}$$

Both equations admit many solutions (in fact, the first equation admits infinitely many). We will start by selecting those solutions whose order (4.21) is minimal. We then obtain

$$\begin{aligned} Q_2^{\mathcal{O}_2|1,1}(y_1;y_1') &= \sigma_1 + \hat{\sigma}_1, && (4.35)\\ Q_2^{\mathcal{O}_2|2,0}(y_1,y_2) &= A_{[0,1]}(\sigma_1^2-3) + A_{[2,0]}K^{[2,0]}, && (4.36) \end{aligned}$$

with $A_{[2,0]}$ constant.

4.2.3 Three-particle form factors

There are three different three-particle form factors that can be obtained from the solutions above: $Q_3^{\mathcal{O}_2|1,2}$, $Q_3^{\mathcal{O}_2|3,0}$ and $Q_3^{\mathcal{O}_2|0,3}$. The first one is the solution of the equations

$$\begin{aligned} Q_3^{\mathcal{O}_2|1,2}(y;-y,y_1) &= (y^2-3)(3+yy_1-y_1^2)A_{[0,1]}, && (4.37)\\ Q_3^{\mathcal{O}_2|1,2}(y_1;y_+,y_-) &= (y^2-3)(A_{[0,1]}((y+y_1)^2-3) + A_{[2,0]}(3-y^2-y_1^2-yy_1)). && (4.38) \end{aligned}$$

The most general solution of these equations with the required order is

$$Q_3^{\mathcal{O}_2|1,2}(y_1;y_1',y_2') = A_{[1,2]}Z^{[1,2]}K^{[0,2]} - \hat{\sigma}_2(A_{[2,0]}-A_{[0,1]})Z^{[1,2]} + A_{[0,1]}(\hat{\sigma}_2\hat{\sigma}_1\sigma_1 - 3K^{[0,2]}), \tag{4.39}$$

which once again involves a new arbitrary constant $A_{[1,2]}$. Let us consider now the three-particle form factor involving only type 1 particles. The equation we need to solve is:

$$Q_3^{\mathcal{O}_2|3,0}(y_+,y_-,y_1) = (y^2-3)(y^2-y_1^2)^2, \tag{4.40}$$

and the most general solution reads

$$Q_3^{\mathcal{O}_2|3,0}(\{y\}_3) = \sigma_1 J^{[3,0]} + A_{[3,0]}K^{[3,0]}, \tag{4.41}$$

with

$$J^{[3,0]} = \sigma_1((3+\sigma_2)^2 - 3\sigma_1^2) - 4\sigma_3(3+\sigma_2), \tag{4.42}$$

and $A_{[3,0]}$ an arbitrary constant. Finally, since the form factor (4.35) is symmetric in its variables, $Q_3^{\mathcal{O}_2|0,3}$ is given by exactly the same function as $Q_3^{\mathcal{O}_2|3,0}$.

4.2.4 Four-particle form factors

There are three four-particle form factors which are related to the solutions above by the boundary and kinematic residue equations: $Q_4^{\mathcal{O}_2|3,1}$, $Q_4^{\mathcal{O}_2|0,4}$ and $Q_4^{\mathcal{O}_2|2,2}$. The polynomial $Q_4^{\mathcal{O}_2|3,1}$ is the solution to

$$Q_4^{\mathcal{O}_2|3,1}(y_1, y_2, -y; y) = Q_2^{\mathcal{O}_2|2,0}(y_1, y_2)P_{0+2}(y, y_1, y_2), \tag{4.43}$$
$$Q_4^{\mathcal{O}_2|3,1}(y_+, y_-, y_1; y_2) = Q_3^{\mathcal{O}_2|1,2}(y_1; y_2, y)W_{1+1}(y, y_1), \tag{4.44}$$

which is given by

$$\begin{aligned} Q_4^{\mathcal{O}_2|3,1}(\{y\}_3; y_1') = {} & (A_{[0,1]} - A_{[2,0]})[\sigma_1\sigma_3(\sigma_1^2(-9+\sigma_2) + 12(3+\sigma_2) - 4\sigma_1\sigma_3)\hat{\sigma}_1 \\ & + J^{[3,0]}(3+\sigma_2)\hat{\sigma}_1^2 - 3K^{[3,0]}(\sigma_1 + 4\hat{\sigma}_1)] - A_{[0,1]}(3Z^{[2,1]}\sigma_1(3\sigma_1^2 - (3+\sigma_2)^2) \\ & + 3\sigma_3(4(3+\sigma_2)^2 + \sigma_1^3\hat{\sigma}_1) + \sigma_3(\sigma_1(9-\sigma_2) + 4\sigma_3)(3(\hat{\sigma}_1 - \sigma_1) + \hat{\sigma}_1(\sigma_2 + \sigma_1\hat{\sigma}_1))) \\ & - (A_{[3,1]}K^{[3,0]} + A_{[1,2]}(K^{[3,0]} + \sigma_1 J^{[3,0]}))Z^{[3,1]}, \end{aligned} \tag{4.45}$$

where $A_{[3,1]}$ is a new arbitrary constant. Let us now compute

$$Q_4^{\mathcal{O}_2|0,4}(y_+, y_-, y_1, y_2) = Q_3^{\mathcal{O}_2|1,2}(y; y_1, y_2)W_{2+0}(y, y_1, y_2), \tag{4.46}$$

which is solved by,

$$\begin{aligned} Q^{\mathcal{O}_2|0,4}(\{y'\}_4) = {} & A_{[0,4]}K^{[0,4]} - A_{[2,0]}\hat{\sigma}_4(-6\hat{\sigma}_1^2 + (3+\hat{\sigma}_2)(9+\hat{\sigma}_2) - 2\hat{\sigma}_1\hat{\sigma}_3 + 2\hat{\sigma}_4)^2 \\ + {} & A_{[0,1]}[3(\hat{\sigma}_1^2(-9+\hat{\sigma}_2) + 12(3+\hat{\sigma}_2) - 4\hat{\sigma}_1\hat{\sigma}_3)((3\hat{\sigma}_1 + \hat{\sigma}_3)^2 - 3(3+\hat{\sigma}_2)^2) \\ & + \hat{\sigma}_4(36\hat{\sigma}_1^4 + (3+\hat{\sigma}_2)^2(-45 + \hat{\sigma}_2(24+\hat{\sigma}_2)) - 3\hat{\sigma}_1^2(9 + \hat{\sigma}_2(36 + 7\hat{\sigma}_2)) \\ & + \hat{\sigma}_3(24\hat{\sigma}_1^3 - 3\hat{\sigma}_1(-1+\hat{\sigma}_2)(9+\hat{\sigma}_2) + 4\hat{\sigma}_3(\hat{\sigma}_1^2 - \hat{\sigma}_2)) \\ & + \hat{\sigma}_4(\hat{\sigma}_2(-3 - 4\hat{\sigma}_1^2 + 7\hat{\sigma}_2) + 4\hat{\sigma}_1\hat{\sigma}_3 - 8(9+\hat{\sigma}_4)))] \\ + {} & A_{[1,2]}[-3(3+\hat{\sigma}_2)(\hat{\sigma}_1^2(-9+\hat{\sigma}_2) + 12(3+\hat{\sigma}_2))(3\hat{\sigma}_1^2 - (3+\hat{\sigma}_2)^2) \\ & - \hat{\sigma}_3(3+\hat{\sigma}_2)(6\hat{\sigma}_1^3(-15+\hat{\sigma}_2) + \hat{\sigma}_1^2(-33+\hat{\sigma}_2)\hat{\sigma}_3 + 12(3+\hat{\sigma}_2)\hat{\sigma}_3) \\ & - 4\hat{\sigma}_1\hat{\sigma}_3(3+\hat{\sigma}_2)(3(3+\hat{\sigma}_2)(9+\hat{\sigma}_2) - \hat{\sigma}_3^2) + 12\hat{\sigma}_1^4\hat{\sigma}_4(-3+\hat{\sigma}_2) \\ & + \hat{\sigma}_4(3+\hat{\sigma}_2)(-6\hat{\sigma}_1^2(3+5\hat{\sigma}_2) + (3+\hat{\sigma}_2)(81 + 42\hat{\sigma}_2 + \hat{\sigma}_2^2)) \\ & - 2\hat{\sigma}_3\hat{\sigma}_4(2\hat{\sigma}_1^3(9-\hat{\sigma}_2) + 3\hat{\sigma}_1(1+\hat{\sigma}_2)(3+\hat{\sigma}_2) + 2(3+\hat{\sigma}_2+2\hat{\sigma}_1^2)\hat{\sigma}_3) \\ & + \hat{\sigma}_4^2(4\hat{\sigma}_1^2(9-\hat{\sigma}_2) + (3+\hat{\sigma}_2)(-15+7\hat{\sigma}_2) + 8(2\hat{\sigma}_1\hat{\sigma}_3 - \hat{\sigma}_4))], \end{aligned} \tag{4.47}$$

where $A_{[0,4]}$ is another arbitrary constant. Finally, we need to solve for the polynomial $Q_4^{\mathcal{O}_2|2,2}$. It is the solution of the following three equations:

$$Q_4^{\mathcal{O}_2|2,2}(y_+, y_-; y_1, y_2) = (y^2-3)Q_3^{\mathcal{O}_2|0,3}(y, y_1, y_2), \tag{4.48}$$
$$Q_4^{\mathcal{O}_2|2,2}(y_1, y_2; y_+, y_-) = (y^2-3)Q_3^{\mathcal{O}_2|3,0}(y, y_1, y_2), \tag{4.49}$$
$$Q_4^{\mathcal{O}_2|2,2}(y, y_1; -y, y_2) = P_{1+1}(y, y_1, y_2)Q_2^{\mathcal{O}_2|1,1}(y_1; y_2). \tag{4.50}$$

Since the polynomials $Q_3^{\mathcal{O}_2|0,3}$ and $Q_3^{\mathcal{O}_2|3,0}$ are identical, the problem is reduced to solving the last two equations and imposing invariance under the transformation $\sigma \leftrightarrow \hat{\sigma}$. The result is

$$\begin{aligned}
Q_4^{\mathcal{O}_2|2,2}(\{y\}_2;\{y'\}_2) &= 9\hat{\sigma}_2(3\sigma_1^2-\sigma_1^4-2\sigma_2)-3\sigma_1\sigma_2(-6\hat{\sigma}_1+2\hat{\sigma}_1\sigma_1^2+\sigma_1\sigma_2)\\
&+ \hat{\sigma}_1^3(\hat{\sigma}_1+2\sigma_1)(-3K^{[2,0]}+\sigma_2^2)+3K^{[2,0]}(3\sigma_1(2\hat{\sigma}_1+\sigma_1)+\sigma_2^2)\\
&+ \hat{\sigma}_1^2(3(K^{[2,0]})^2+3K^{[2,0]}\sigma_2+\sigma_2^3)+\hat{\sigma}_2^2(9+\hat{\sigma}_1{\sigma_1}^3+(3+\hat{\sigma}_1^2)\sigma_2)\\
&- 3\sigma_1\hat{\sigma}_2^2(1+\sigma_2)(\hat{\sigma}_1+\sigma_1)+\hat{\sigma}_2(3(3\sigma_1^2-2\sigma_2)\sigma_2-\hat{\sigma}_1\sigma_1^3(12-\sigma_2))\\
&+ \hat{\sigma}_2(3(4-\sigma_2)(3+\sigma_2)+\hat{\sigma}_1^2(9-\sigma_1^2(3-5\sigma_2)-\sigma_2(6+5\sigma_2-2\hat{\sigma}_1\sigma_1)))\\
&+ K^{[2,0]}Z^{[2,2]}(A_{[2,2]}K^{[2,0]}-3(2-A_{[3,0]})-\hat{\sigma}_2(1-A_{[3,0]}))\\
&- Z^{[2,2]}A_{[3,0]}(9-3\sigma_1^2+\hat{\sigma}_1^2\sigma_2-\hat{\sigma}_2\sigma_2),
\end{aligned} \tag{4.51}$$

with $A_{[2,2]}$ constant.

4.2.5 Remarks on operator identification

In the previous subsection, all our form factors have "descended" either from $Q_1^{\mathcal{O}_2|0,1}$ or from $Q_0^{\mathcal{O}_2|0,0}$, so that we had no need to fix $Q_1^{\mathcal{O}_2|1,0}$. One obvious option is to choose $Q_1^{\mathcal{O}_2|0,1} = Q_1^{\mathcal{O}_2|1,0}$ and, more generally, to consider form factor solutions which are completely symmetric under the exchange of the names of the particles. Such form factors should correspond to a particular type of operators which we will call "symmetric". We mean by this, that they should be symmetric on the fundamental fields, such as $\mathcal{O}_{12} = \phi_1 + \phi_2$ and $\hat{\mathcal{O}}_{12} = \phi_1\partial_x\phi_2 + \phi_2\partial_x\phi_1$. But there are of course many fields which are not symmetric, for example those that depend only either on ϕ_1 ("type-1 fields") or on ϕ_2 ("type-2 fields"). It is natural to assume that for any such fields the one particle form factors $Q_1^{\mathcal{O}_2|0,1}$ (for type-1) or $Q_1^{\mathcal{O}_2|1,0}$ (for type 2) should vanish. Therefore, the solutions which we just constructed could correspond to a "type-2 field" $\mathcal{O}_2$ if we impose the condition that all form factors descending from $Q_1^{\mathcal{O}_2|1,0}$ are vanishing. In order to identify the precise operator our solutions correspond to we should analyze the UV behaviour of the two-point function. We can however get some idea by just analyzing the order of the form factors obtained. As anticipated before

$$[F_{m+n}^{\mathcal{O}_2|m,n}] = 0. \tag{4.52}$$

This indicates that the boundary field to which the solutions correspond to is a spinless field in the bulk theory. If we assume, as mentioned above, that this field has $F_1^{\mathcal{O}_2|1,0} = 0$, then a natural candidate is the fundamental field ϕ_2 and any powers thereof. This constitutes an infinite countable set of fields, in correspondence with the infinite countable set of solutions which the form factor equations seem to have. We should now provide some more reasoning to justify our last statement: notice that in all our examples each new form factor involves a new arbitrary constant. If we assume that this phenomenon will continue for higher particle form factors, then the full set of solutions to our equations will depend on a infinite but countable number of arbitrary parameters. One way of understanding this is to assume that the solutions obtained correspond in fact to a field which is a linear combination of an infinite but countable set of "type-2" fields, in one-to-one correspondence with the constants $A_{[m,n]}$ (since the form factor of a sum of fields equals the sum of their form factors).

4.2.6 From boundary to bulk form factors

It is easy to see that from every boundary form factor solution, a bulk form factor solution can be obtained by the simple procedure of shifting all rapidities to infinity and selecting out the leading order terms [25]. A way to see this is to notice that when such shift is performed, the boundary form factor consistency equations tend to the bulk equations because of the properties

$$\lim_{\theta\to\infty} R_i(\theta) = \lim_{\theta\to\infty} w_i(\theta) = \lim_{\theta\to\infty} S_{ij}(\theta) = \lim_{\theta\to\infty} f_{ij}(\theta) = 1. \tag{4.53}$$

These imply that equations (3.2) and (3.3) become equivalent to each other and equal to the standard crossing relation for the bulk form factors. At the same time, the S-matrix product depending on the sum of rapidities becomes 1 in (3.4) rendering it in its bulk form [2, 3].

It is therefore interesting to consider the limit described above for the solutions we have just obtained

$$Q^{\mathcal{O}_2|0,1}_{1;\text{bulk}}(y'_1) = A_{[0,1]}, \tag{4.54}$$
$$Q^{\mathcal{O}_2|2,0}_{2;\text{bulk}}(\{y\}_2) = A_{[0,1]}\sigma_1^2 + A_{[2,0]}K^{[2,0]}_{\text{bulk}}, \tag{4.55}$$
$$Q^{\mathcal{O}_2|1,1}_{2;\text{bulk}}(y_1; y'_1) = \sigma_1 + \hat\sigma_1, \tag{4.56}$$
$$Q^{\mathcal{O}_2|1,2}_{3;\text{bulk}}(y_1; y'_1, y'_2) = (A_{[1,2]}K^{[0,2]}_{\text{bulk}} - \hat\sigma_2(A_{[2,0]} - A_{[0,1]}))Z^{[1,2]} + A_{[0,1]}\hat\sigma_2\hat\sigma_1\sigma_1, \tag{4.57}$$
$$Q^{\mathcal{O}_2|3,0}_{3;\text{bulk}}(\{y\}_3) = \sigma_1\sigma_2(\sigma_1\sigma_2 - 4\sigma_3) + A_{[3,0]}K^{[3,0]}_{\text{bulk}}, \tag{4.58}$$
$$\begin{aligned} Q^{\mathcal{O}_2|3,1}_{4;\text{bulk}}(\{y\}_3; y'_1) &= \hat\sigma_1(A_{[0,1]} - A_{[2,0]})(\sigma_1^2\sigma_3 + \sigma_2^2\hat\sigma_1)(\sigma_1\sigma_2 - 4\sigma_3) - A_{[3,1]}K^{[3,0]}_{\text{bulk}}Z^{[3,1]} \\ &\quad - A_{[1,2]}K^{[3,0]}_{\text{bulk}}Z^{[3,1]} + (\sigma_1\sigma_2 - 4\sigma_3)(A_{[0,1]}\hat\sigma_1\sigma_3(\sigma_2 + \sigma_1\hat\sigma_1) - A_{[1,2]}\sigma_1\sigma_2 Z^{[3,1]}), \end{aligned} \tag{4.59}$$
$$\begin{aligned} Q^{\mathcal{O}_2|0,4}_{4;\text{bulk}}(\{y'\}_4) &= A_{[0,4]}K^{[0,4]}_{\text{bulk}} - A_{[2,0]}\hat\sigma_4(\hat\sigma_2^2 - 2\hat\sigma_1\hat\sigma_3 + 2\hat\sigma_4)^2 \\ &\quad + A_{[0,1]}\sigma_4[4(\sigma_1\sigma_3 - \sigma_4)(\sigma_1\sigma_3 + 2\sigma_4) + \sigma_2^2(\sigma_2^2 - 3\sigma_1\sigma_3 + 7\sigma_4) - 4\sigma_2(\sigma_3^2 + \sigma_1^2\sigma_4)] \\ &\quad + A_{[1,2]}[-\sigma_1\sigma_2\sigma_3^2(\sigma_1\sigma_2 - 4\sigma_3) + \sigma_2^2\sigma_4(\sigma_2^2 - 6\sigma_1\sigma_3) + (4\sigma_2(\sigma_1^3 - \sigma_3)\sigma_3 - 8\sigma_1^2\sigma_3^2)\sigma_4 \\ &\quad + (7\sigma_2^2 - 4\sigma_1(\sigma_1\sigma_2 - 4\sigma_3))\sigma_4^2 - 8\sigma_4^3], \end{aligned} \tag{4.60}$$
$$\begin{aligned} Q^{\mathcal{O}_2|2,2}_{4;\text{bulk}}(\{y\}_2; \{y'\}_2) &= \hat\sigma_1\hat\sigma_2\sigma_2(2\hat\sigma_1^2\sigma_1 + \sigma_1(\sigma_1^2 - 3\sigma_2) - 5\hat\sigma_1 K^{[2,0]}_{\text{bulk}}) \\ &\quad + \sigma_2^2(\hat\sigma_1^2(\hat\sigma_1(\hat\sigma_1 + 2\sigma_1) + \sigma_2) + \sigma_1^2(\hat\sigma_1\sigma_1 - 3\sigma_2) + \hat\sigma_1(\hat\sigma_1 - 3\sigma_1)\sigma_2) \\ &\quad + Z^{[2,2]}((A_{[2,2]}K^{[0,2]}_{\text{bulk}} - \hat\sigma_2)K^{[2,0]}_{\text{bulk}} + A_{[3,0]}(\hat\sigma_2 K^{[2,0]}_{\text{bulk}} + \sigma_2 K^{[0,2]}_{\text{bulk}})). \end{aligned} \tag{4.61}$$

We can now compare these solutions to the ones obtained in [32]. It turns out that, up to the three-particle form factors, they completely agree with those when setting $q = 1$ (which is equivalent to $B = 1$ in our notation) and matching the constants appropriately. There is also agreement for the four-particle form factor (4.59) but there are some important differences between our solutions and those of T. Oota for other four-particle form factors, even though their orders agree. The precise differences are:

$$Q^{\mathcal{O}_2|0,4}_{4;\text{bulk}}(\{y'\}_4) - Q^{\mathcal{O}_2|0,4}_{4;\text{Oota}}(\{y'\}_4) = 2A_{[0,1]}\sigma_4^2(\sigma_1\sigma_3 - \sigma_4) + (A_{[0,1]} - A_{[2,0]})\sigma_2^2\sigma_4(\sigma_2^2 + 2\sigma_4), \tag{4.62}$$

and

$$\begin{aligned} Q^{\mathcal{O}_2|2,2}_{4;\text{bulk}}(\{y\}_2; \{y'\}_2) - Q^{\mathcal{O}_2|2,2}_{4;\text{Oota}}(\{y\}_2; \{y'\}_2) &= 3Z^{[2,2]}(1 - A_{[3,0]})(K^{[2,0]} - \hat\sigma_1^2) \\ &\quad + \hat\sigma_1^2\hat\sigma_2(\sigma_1^2 + \sigma_2)(3 - K^{[2,0]}) - \hat\sigma_2\sigma_1(\hat\sigma_1\sigma_2(\hat\sigma_2 + \sigma_2) + \sigma_1(\hat\sigma_1^2\sigma_1^2 + \hat\sigma_2\sigma_2)). \end{aligned} \tag{4.63}$$

We believe that Oota's solutions for these two cases must be wrong, as the agreement of all other form factors with ours strongly suggests that we are dealing with the same class of fields.

4.3 Form factors of "spin-1" fields

In this section we want to find further solutions to the boundary form factors equations and, as a by-product, also new solutions to the bulk form factor equations. We will characterize those solutions by the choice

$$Q_1^{\hat{\mathcal{O}}_2|0,1}(y) = B_{[0,1]} y, \tag{4.64}$$

where $B_{[0,1]}$ is an arbitrary constant. In addition, we would like our form factors to have

$$[F_{m+n}^{\hat{\mathcal{O}}_2|m,n}] = 1, \qquad \Leftrightarrow \qquad [Q_{m+n}^{\hat{\mathcal{O}}_2|m,n}] = m^2 + (m+n)(n-1) + 1, \tag{4.65}$$

such that the field $\hat{\mathcal{O}}_2$ can be matched to a spin-1 field of the bulk theory. If we choose $Q_1^{\hat{\mathcal{O}}_2|1,0} = 0$, we can identify this field as being of "type-2" again (like $\partial_x \phi_2$). An obvious solution to these constraints is obtained by multiplying all solutions of the previous subsection by the polynomial $\sigma_1 + \hat{\sigma}_1$. If we do so we can identify $\hat{\mathcal{O}}_2 = \partial_x \mathcal{O}_2$, but other less trivial solutions also exist, as we show below.

4.3.1 Two-particle form factors

A set of solutions, which is compatible with (4.64) is

$$\begin{aligned} Q_2^{\hat{\mathcal{O}}_2|1,1}(y_1, y_1') &= \hat{\sigma}_1(\sigma_1 + \hat{\sigma}_1), && (4.66)\\ Q_2^{\hat{\mathcal{O}}_2|2,0}(y_1, y_2) &= B_{[0,1]}\sigma_1(\sigma_1^2 - 3) + (\hat{A}_{[2,0]} + B_{[2,0]}\sigma_1)K^{[2,0]}, && (4.67) \end{aligned}$$

where $\hat{A}_{[2,0]}$ and $B_{[2,0]}$ are constants.

4.3.2 Three-particle form factors

The three-particle form factors are given by

$$\begin{aligned} Q_3^{\hat{\mathcal{O}}_2|1,2}(y_1, y_1', y_2') &= (\hat{A}_{[1,2]} + (B_{[1,2]} - B_{[2,0]})\sigma_1 + C_{[1,2]}\hat{\sigma}_1)Z^{[1,2]}K^{[0,2]} \\ &\quad -\hat{\sigma}_2(\hat{A}_{[2,0]} + B_{[2,0]}(\sigma_1 + \hat{\sigma}_1))Z^{[1,2]} - B_{[0,1]}[3K^{[0,2]}\hat{\sigma}_1 + \sigma_1^2\hat{\sigma}_1(-6 + 2\hat{\sigma}_1^2 - 5\hat{\sigma}_2) \\ &\quad +\sigma_1^3(-6 + 2\hat{\sigma}_1^2 - 3\hat{\sigma}_2) - \hat{\sigma}_1\hat{\sigma}_2^2 - 3\sigma_1(-3 + \hat{\sigma}_1^2 + \hat{\sigma}_2 + \hat{\sigma}_2^2)], && (4.68)\\ Q_3^{\hat{\mathcal{O}}_2|3,0}(\{y\}_3) &= (\sigma_2 + 3)J^{[3,0]} + \sigma_1\hat{A}_{[3,0]}K^{[3,0]}, && (4.69)\\ Q_3^{\hat{\mathcal{O}}_2|0,3}(\{y'\}_3) &= \hat{\sigma}_3(\hat{\sigma}_1^2 - 2(3 + \hat{\sigma}_2))^2 + (B_{[0,3]} + C_{[0,3]}\hat{\sigma}_1)K^{[0,3]}, && (4.70) \end{aligned}$$

where all variables $\hat{A}_{[a,b]}$, $B_{[a,b]}$ and $C_{[a,b]}$ are constants. Notice that, since (4.66) is not anymore a symmetric function, the polynomials $Q_3^{\hat{\mathcal{O}}_2|3,0}$ and $Q_3^{\hat{\mathcal{O}}_2|0,3}$ are now different from each other.

4.3.3 Four-particle form factors

The four-particle form factors are given by

$$
\begin{aligned}
Q_4^{\hat{\mathcal{O}}_2|3,1}(\{y\}_3, y_1') = {} & (\hat{A}_{[3,1]} + B_{[3,1]}\sigma_1 + C_{[3,1]}\hat{\sigma}_1)K^{[3,0]}Z^{[3,1]} \\
& -\hat{A}_{[1,2]}((3+\sigma_2)(-3\sigma_1^2 + (3+\sigma_2)^2) + \sigma_1(\sigma_1^2 - 4(3+\sigma_2))\sigma_3)Z^{[3,1]} \\
& -(B_{[1,2]} - B_{[2,0]})(-4K^{[3,0]}\hat{\sigma}_1 + (\sigma_1^2 - 2(3+\sigma_2))^2\sigma_3)Z^{[3,1]} \\
& -C_{[1,2]}((3+\sigma_2)(J^{[3,0]} - \hat{\sigma}_1(3\sigma_1^2 - (3+\sigma_2)^2)) + \hat{\sigma}_1\sigma_1(\sigma_1^2 - 4(3+\sigma_2))\sigma_3)Z^{[3,1]} \\
& +B_{[2,0]}(\hat{\sigma}_1 + \sigma_1)[3K^{[3,0]}(4\hat{\sigma}_1 + \sigma_1) - J^{[3,0]}\hat{\sigma}_1^2(3+\sigma_2) - 12\hat{\sigma}_1\sigma_1\sigma_3K^{[2,0]} \\
& -\hat{\sigma}_1\sigma_1\sigma_3(\sigma_1^2(3+\sigma_2) - 4\sigma_1\sigma_3)] + B_{[0,1]}[9K^{[3,0]}\hat{\sigma}_1^4 + 3\sigma_1^2(3-\sigma_1^2)(3\sigma_1^2 - (3+\sigma_2)^2) \\
& -3\hat{\sigma}_1\sigma_1(18\sigma_1^4 + (3+\sigma_2)^2(15+4\sigma_2) - 3\sigma_1^2(33+16\sigma_2+2\sigma_2^2)) \\
& +\hat{\sigma}_1^2(3(3+\sigma_2)^3(-4+3\sigma_2) + 3\sigma_1^4(-21+8\sigma_2) + \sigma_1^2(297+9\sigma_2-54\sigma_2^2-8\sigma_2^3)) \\
& -3\sigma_1(\sigma_1^4 + \sigma_1^2(-9+\sigma_2) + 4(3+\sigma_2)^2)\sigma_3 - 2(\sigma_1^4 + 4(3+\sigma_2)^2 - 2\sigma_1^2(9+2\sigma_2))\sigma_3^2 \\
& +\hat{\sigma}_1\sigma_3(-\sigma_1^4\sigma_2 + (3+\sigma_2)^2(15+\sigma_2) - 18\sigma_1^2(7+3\sigma_2) + \sigma_1^3\sigma_3(5\sigma_1^2 - 4\sigma_2)) \\
& -\hat{\sigma}_1^2\sigma_3(2\sigma_1^5 + \sigma_1^3(9-19\sigma_2) + 11\sigma_1(3+\sigma_2)^2 + 8\sigma_1^2\sigma_3 + 4(3+\sigma_2)\sigma_3) \\
& +\hat{\sigma}_1^3(\sigma_1(-33+9\sigma_1^2-10\sigma_2)(3\sigma_1^2 - (3+\sigma_2)^2) \\
& +\sigma_3(7\sigma_1^4 - 12(3+\sigma_2)^2 + 3\sigma_1^2(5+3\sigma_2) - 4\sigma_1\sigma_3))],
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
Q_4^{\hat{\mathcal{O}}_2|2,2}(\{y\}_2, \{y'\}_2) = {} & 9\hat{\sigma}_1(K^{[0,2]}\hat{\sigma}_1^2 - K^{[0,2]}\hat{\sigma}_2 - \hat{\sigma}_2^2)9K^{[0,2]}(2\hat{\sigma}_1^2 - 3\hat{\sigma}_2)\sigma_1 \\
& -3\hat{\sigma}_2(-18 + 6\hat{\sigma}_1^2 + \hat{\sigma}_2^2)\sigma_1 + 3K^{[0,2]}\hat{\sigma}_1(K^{[0,2]} - \hat{\sigma}_2)\sigma_1^2 \\
& +(-6K^{[0,2]}\hat{\sigma}_1^2 + \hat{\sigma}_2^2(6+\hat{\sigma}_2))\sigma_1^3 + \hat{\sigma}_1(-3K^{[0,2]} + \hat{\sigma}_2(K^{[0,2]} + \hat{\sigma}_2))\sigma_1^4 \\
& +K^{[0,2]}\hat{\sigma}_2\sigma_1^5 + \sigma_1(27 + 27\hat{\sigma}_2 + 9\hat{\sigma}_2^2 + 4\hat{\sigma}_2^3 - 9\sigma_1^2 - 9\hat{\sigma}_2\sigma_1^2 - \hat{\sigma}_2^2\sigma_1^2) \\
& +\hat{\sigma}_1(9K^{[0,2]} + 3(3\hat{\sigma}_1^2 + \hat{\sigma}_2^2) - (9 + 6\hat{\sigma}_2 + 4\hat{\sigma}_2^2)\sigma_1^2 + (K^{[0,2]} + \hat{\sigma}_1^2)\sigma_1^4) \\
& +\hat{\sigma}_1^3(-(\sigma_1^2(-3+\sigma_1^2)) + 3\hat{\sigma}_2(-1+\sigma_1^2)) - \hat{\sigma}_1^4(3\hat{\sigma}_1 + \sigma_1(3 - \hat{\sigma}_2 + \sigma_1^2)) \\
& +\hat{\sigma}_1^2\sigma_1(-5\hat{\sigma}_2^2 + 6\sigma_1^2 + \hat{\sigma}_2(-12+5\sigma_1^2))\sigma_2 + \sigma_2^2(-3\hat{\sigma}_1(2 + \hat{\sigma}_1^2 - \hat{\sigma}_2)\hat{\sigma}_2) \\
& +\sigma_2^2(-\hat{\sigma}_2\sigma_1(6 + 5\hat{\sigma}_2 - 3\hat{\sigma}_1\sigma_1) + K^{[0,2]}(3\hat{\sigma}_1 + \sigma_1^3) - \hat{\sigma}_2(3\hat{\sigma}_1 - \sigma_1)\sigma_2) \\
& +K^{[0,2]}Z^{[2,2]}[(-1 + \hat{A}_{[3,0]} + C_{[0,3]})\sigma_1(-3+\sigma_1^2) + (B_{[0,3]} + C_{[0,3]}\hat{\sigma}_1)\sigma_2] \\
& -Z^{[2,2]}[K^{[0,2]}K^{[2,0]}(\hat{A}_{[2,2]} + B_{[2,2]}\hat{\sigma}_1) - \hat{A}_{[3,0]}(K^{[0,2]}\sigma_1(3-\sigma_1^2) + \hat{\sigma}_2(\hat{\sigma}_1 + \sigma_1)K^{[2,0]})],
\end{aligned}
\tag{4.72}
$$

and

$$
\begin{aligned}
Q_4^{\hat{\mathcal{O}}_2|0,4}(\{y\}_3, y_1') = {}& (\hat{A}_{[0,4]} + B_{[0,4]}\hat{\sigma}_1)K^{[0,4]} \\
& -(\hat{A}_{[2,0]} + B_{[1,2]}\hat{\sigma}_1)\hat{\sigma}_4(27 - 6\hat{\sigma}_1^2 + 12\hat{\sigma}_2 + \hat{\sigma}_2^2 - 2\hat{\sigma}_1\hat{\sigma}_3 + 2\hat{\sigma}_4)^2 \\
& +B_{[0,1]}[3\hat{\sigma}_1(9\hat{\sigma}_1^4(15 + \hat{\sigma}_2) + 2(3 + \hat{\sigma}_2)^3(27 + \hat{\sigma}_2(6 + \hat{\sigma}_2)) \\
& -9\hat{\sigma}_1^2(3 + \hat{\sigma}_2)(33 + \hat{\sigma}_2(10 + \hat{\sigma}_2))) + 6(-4(3 + \hat{\sigma}_2)^4 + 3\hat{\sigma}_1^4(21 + \hat{\sigma}_2))\hat{\sigma}_3 \\
& -12\hat{\sigma}_1^2(54 + 27\hat{\sigma}_2 + 6\hat{\sigma}_2^2 + \hat{\sigma}_2^3)\hat{\sigma}_3 + \hat{\sigma}_1(270 + 198\hat{\sigma}_2 + 30\hat{\sigma}_2^2 - 2\hat{\sigma}_2^3 + 3\hat{\sigma}_1^2(39 + \hat{\sigma}_2))\hat{\sigma}_3^2 \\
& +4(3\hat{\sigma}_1^2 + 2(3 + \hat{\sigma}_2)^2)\hat{\sigma}_3^3 + 3\hat{\sigma}_1(36\hat{\sigma}_1^4 - \hat{\sigma}_1^2(585 + \hat{\sigma}_2(180 + 7\hat{\sigma}_2)))\hat{\sigma}_4 \\
& +3\hat{\sigma}_1(3 + \hat{\sigma}_2)(567 + \hat{\sigma}_2(231 + \hat{\sigma}_2(25 + \hat{\sigma}_2)))\hat{\sigma}_4 + 4\hat{\sigma}_1(-30 + 3\hat{\sigma}_1^2 - 7\hat{\sigma}_2)\hat{\sigma}_3^2\hat{\sigma}_4 \\
& +3(234 + 24\hat{\sigma}_1^4 + 114\hat{\sigma}_2 + 6\hat{\sigma}_2^2 - 2\hat{\sigma}_2^3 - \hat{\sigma}_1^2(303 + 80\hat{\sigma}_2 + \hat{\sigma}_2^2))\hat{\sigma}_3\hat{\sigma}_4 \\
& +(-4\hat{\sigma}_1^3(12 + \hat{\sigma}_2) + 3\hat{\sigma}_1(168 + 63\hat{\sigma}_2 + 5\hat{\sigma}_2^2) - 12\hat{\sigma}_1^2\hat{\sigma}_3 + 24(4 + \hat{\sigma}_2)\hat{\sigma}_3)\hat{\sigma}_4^2] \\
& +\hat{A}_{[1,2]}[3(-9\hat{\sigma}_1^6 - 18\hat{\sigma}_1^2(3 + \hat{\sigma}_2)^2(5 + \hat{\sigma}_2) + (3 + \hat{\sigma}_2)^4(15 + \hat{\sigma}_2) + 3\hat{\sigma}_1^4(54 + 21\hat{\sigma}_2 + \hat{\sigma}_2^2)) \\
& -3\hat{\sigma}_1(9\hat{\sigma}_1^4 + 6(3 + \hat{\sigma}_2)^2(7 + \hat{\sigma}_2) - \hat{\sigma}_1^2(135 + 48\hat{\sigma}_2 + \hat{\sigma}_2^2))\hat{\sigma}_3 \\
& -(9\hat{\sigma}_1^4 - 39\hat{\sigma}_1^2(3 + \hat{\sigma}_2) + (3 + \hat{\sigma}_2)^2(15 + \hat{\sigma}_2))\hat{\sigma}_3^2 + \hat{\sigma}_1(-\hat{\sigma}_1^2 + 4(3 + \hat{\sigma}_2))\hat{\sigma}_3^3 \\
& +(3\hat{\sigma}_1^4(-3 + 4\hat{\sigma}_2) + 2(3 + \hat{\sigma}_2)^2(36 + 21\hat{\sigma}_2 + \hat{\sigma}_2^2) - \hat{\sigma}_1^2(135 + 153\hat{\sigma}_2 + 39\hat{\sigma}_2^2 + \hat{\sigma}_2^3))\hat{\sigma}_4 \\
& +\hat{\sigma}_1(-27 - 24\hat{\sigma}_2 - 5\hat{\sigma}_2^2 + 2\hat{\sigma}_1^2(-9 + 2\hat{\sigma}_2))\hat{\sigma}_3\hat{\sigma}_4 - (5\hat{\sigma}_1^2 + 4(3 + \hat{\sigma}_2))\hat{\sigma}_3^2\hat{\sigma}_4 \\
& +(-54 + \hat{\sigma}_1^2(27 - 4\hat{\sigma}_2) - 3\hat{\sigma}_2 + 5\hat{\sigma}_2^2 + 13\hat{\sigma}_1\hat{\sigma}_3 - 7\hat{\sigma}_4)\hat{\sigma}_4^2] \\
& +(B_{[2,0]} - B_{[1,2]})[-3\hat{\sigma}_1(108\hat{\sigma}_1^4 - 3\hat{\sigma}_1^2(3 + \hat{\sigma}_2)(9 + \hat{\sigma}_2)^2 + (3 + \hat{\sigma}_2)^3(45 + \hat{\sigma}_2(6 + \hat{\sigma}_2))) \\
& +6(-54\hat{\sigma}_1^4 + 2(3 + \hat{\sigma}_2)^4 + \hat{\sigma}_1^2(3 + \hat{\sigma}_2)(45 + \hat{\sigma}_2(6 + \hat{\sigma}_2)))\hat{\sigma}_3 + 4\hat{\sigma}_1(-2\hat{\sigma}_1^2 + 3(5 + \hat{\sigma}_2))\hat{\sigma}_3^2\hat{\sigma}_4 \\
& +\hat{\sigma}_1(-108\hat{\sigma}_1^2 + (3 + \hat{\sigma}_2)(-27 + (-18 + \hat{\sigma}_2)\hat{\sigma}_2))\hat{\sigma}_3^2 - 4(3\hat{\sigma}_1^2 + (3 + \hat{\sigma}_2)^2)\hat{\sigma}_3^3 \\
& -\hat{\sigma}_1(3483 + 72\hat{\sigma}_1^4 + 2565\hat{\sigma}_2 + 603\hat{\sigma}_2^2 + 51\hat{\sigma}_2^3 + 2\hat{\sigma}_2^4 - 12\hat{\sigma}_1^2(99 + 30\hat{\sigma}_2 + \hat{\sigma}_2^2))\hat{\sigma}_4 \\
& +(-48\hat{\sigma}_1^4 + 4\hat{\sigma}_1^2(144 + 39\hat{\sigma}_2 + \hat{\sigma}_2^2) + 3(-117 - 57\hat{\sigma}_2 - 3\hat{\sigma}_2^2 + \hat{\sigma}_2^3))\hat{\sigma}_3\hat{\sigma}_4 \\
& -4\hat{\sigma}_4^2(-12\hat{\sigma}_1^3 - 4\hat{\sigma}_1^2\hat{\sigma}_3 + 3(4 + \hat{\sigma}_2)\hat{\sigma}_3 + \hat{\sigma}_1(99 + 2\hat{\sigma}_2(18 + \hat{\sigma}_2) + 2\hat{\sigma}_4))] \\
& +C_{[1,2]}[-9\hat{\sigma}_1(3\hat{\sigma}_1^6 - 4\hat{\sigma}_2(3 + \hat{\sigma}_2)^3 - \hat{\sigma}_1^4(18 + 21\hat{\sigma}_2 + \hat{\sigma}_2^2) + \hat{\sigma}_1^2(27 + 99\hat{\sigma}_2 + 45\hat{\sigma}_2^2 + 5\hat{\sigma}_2^3)) \\
& -3(9\hat{\sigma}_1^6 - 4(3 + \hat{\sigma}_2)^4 - \hat{\sigma}_1^4(27 + 48\hat{\sigma}_2 + \hat{\sigma}_2^2) + 4\hat{\sigma}_1^2(27 + 45\hat{\sigma}_2 + 15\hat{\sigma}_2^2 + \hat{\sigma}_2^3))\hat{\sigma}_3 \\
& -3\hat{\sigma}_1(3\hat{\sigma}_1^4 - \hat{\sigma}_1^2(3 + 13\hat{\sigma}_2) + 12(6 + 5\hat{\sigma}_2 + \hat{\sigma}_2^2))\hat{\sigma}_3^2 - (\hat{\sigma}_1^4 - 4\hat{\sigma}_1^2\hat{\sigma}_2 + 4(3 + \hat{\sigma}_2)^2)\hat{\sigma}_3^3 \\
& +\hat{\sigma}_1(3\hat{\sigma}_1^4(-15 + 4\hat{\sigma}_2) - \hat{\sigma}_1^2(-729 - 63\hat{\sigma}_2 + 39\hat{\sigma}_2^2 + \hat{\sigma}_2^3))\hat{\sigma}_4 \\
& +\hat{\sigma}_1(3 + \hat{\sigma}_2)(-702 + \hat{\sigma}_2(-135 + \hat{\sigma}_2(24 + \hat{\sigma}_2)))\hat{\sigma}_4 + \hat{\sigma}_1(-9\hat{\sigma}_1^2 + 8(6 + \hat{\sigma}_2))\hat{\sigma}_3^2\hat{\sigma}_4 \\
& +(\hat{\sigma}_1^4(-42 + 4\hat{\sigma}_2) + \hat{\sigma}_1^2(441 + 84\hat{\sigma}_2 - 5\hat{\sigma}_2^2) + 3(-117 - 57\hat{\sigma}_2 - 3\hat{\sigma}_2^2 + \hat{\sigma}_2^3))\hat{\sigma}_3\hat{\sigma}_4 \\
& -\hat{\sigma}_4^2(\hat{\sigma}_1^3(-51 + 4\hat{\sigma}_2) - 21\hat{\sigma}_1^2\hat{\sigma}_3 + 12(4 + \hat{\sigma}_2)\hat{\sigma}_3 + \hat{\sigma}_1(342 + 99\hat{\sigma}_2 - \hat{\sigma}_2^2 + 11\hat{\sigma}_4))].
\end{aligned} \tag{4.73}
$$

4.3.4 From boundary to bulk form factors

The bulk counterparts of the solutions above are:

$$
\begin{aligned}
Q_{2;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|1,1}(y_1,y_1') &= \hat{\sigma}_1(\sigma_1+\hat{\sigma}_1), && (4.74)\\
Q_{2;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|2,0}(y_1,y_2) &= \sigma_1(B_{[0,1]}\sigma_1^2+B_{[2,0]}K_{\mathrm{bulk}}^{[2,0]}), && (4.75)\\
Q_{3;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|1,2}(y_1,y_1',y_2') &= [(B_{[1,2]}-B_{[2,0]})\sigma_1+C_{[1,2]}\hat{\sigma}_1]Z^{[1,2]}K_{\mathrm{bulk}}^{[0,2]}-\hat{\sigma}_2B_{[2,0]}(\sigma_1+\hat{\sigma}_1)Z^{[1,2]} \\
&+ B_{[0,1]}[(\sigma_1+\hat{\sigma}_1)(\hat{\sigma}_2^2-2\sigma_1^2\hat{\sigma}_1^2+3\hat{\sigma}_2\sigma_1^2)+2\hat{\sigma}_2\sigma_1(\hat{\sigma}_2+\sigma_1\hat{\sigma}_1)], && (4.76)\\
Q_{3;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|3,0}(\{y\}_3) &= \sigma_2^2(\sigma_1\sigma_2-4\sigma_3)+\sigma_1\hat{A}_{[3,0]}K_{\mathrm{bulk}}^{[3,0]}, && (4.77)\\
Q_{3;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|0,3}(\{y'\}_3) &= 4\hat{\sigma}_2^2\hat{\sigma}_3+\hat{\sigma}_1C_{[0,3]}K_{\mathrm{bulk}}^{[0,3]}, && (4.78)\\
Q_{4;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|3,1}(\{y\}_3,y_1') &= (B_{[3,1]}\sigma_1+C_{[3,1]}\hat{\sigma}_1)K_{\mathrm{bulk}}^{[3,0]}Z^{[3,1]} \\
&+ (B_{[2,0]}-B_{[1,2]})((\sigma_1^2-2\,\sigma_2)^2\sigma_3-4K_{\mathrm{bulk}}^{[3,0]}\hat{\sigma}_1)Z^{[3,1]} \\
&- C_{[1,2]}(\sigma_2^2(\sigma_1\sigma_2-4\sigma_3)+\sigma_2^3\hat{\sigma}_1+\sigma_1(\sigma_1^2-4\sigma_2)\sigma_3\hat{\sigma}_1)Z^{[3,1]} \\
&- B_{[2,0]}\hat{\sigma}_1(\sigma_1+\hat{\sigma}_1)(\sigma_1\sigma_2-4\sigma_3)(\sigma_1^2\sigma_3+\sigma_2^2\hat{\sigma}_1) \\
&+ B_{[0,1]}[-2(\sigma_1^2-2\sigma_2)^2\sigma_3^2+\sigma_3(-\sigma_1^4\sigma_2+\sigma_2^3+\sigma_1(5\sigma_1^2-4\sigma_2)\sigma_3)\hat{\sigma}_1 && (4.79)\\
&+ (\sigma_1\sigma_2^3(9\sigma_2-8\sigma_1^2)(19\sigma_1^2\sigma_2-11\sigma_2^2-2\sigma_1^4)\sigma_3-4(2\sigma_1^2+\sigma_2)\sigma_3^2)\hat{\sigma}_1^2 \\
&+ (\sigma_1\sigma_2^2(10\sigma_2-9\sigma_1^2)+\sigma_3(7\sigma_1^4+9\sigma_1^2\sigma_2-12\sigma_2^2-4\sigma_1\sigma_3)+9K_{\mathrm{bulk}}^{[3,0]}\hat{\sigma}_1)\hat{\sigma}_1^3], \\
Q_{4;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|0,4}(\{y'\}_4) &= B_{[0,4]}\hat{\sigma}_1K_{\mathrm{bulk}}^{[0,4]}-B_{[1,2]}\hat{\sigma}_1\hat{\sigma}_4(\hat{\sigma}_2^2-2\hat{\sigma}_1\hat{\sigma}_3+2\hat{\sigma}_4)^2 \\
&+ (B_{[2,0]}-B_{[1,2]})[\hat{\sigma}_2^2(\hat{\sigma}_1\hat{\sigma}_2-4\hat{\sigma}_3)\hat{\sigma}_3^2-2\hat{\sigma}_1\hat{\sigma}_2^4\hat{\sigma}_4+\hat{\sigma}_2^2(4\hat{\sigma}_1^2+3\hat{\sigma}_2)\hat{\sigma}_3\hat{\sigma}_4 \\
&- 4\hat{\sigma}_1(2\hat{\sigma}_1^2-3\hat{\sigma}_2)\hat{\sigma}_3^2\hat{\sigma}_4+4(4\hat{\sigma}_1^2-3\hat{\sigma}_2)\hat{\sigma}_3\hat{\sigma}_4^2-8\hat{\sigma}_1(\hat{\sigma}_2^2+\hat{\sigma}_4)\hat{\sigma}_4^2] \\
&+ B_{[0,1]}[-2\hat{\sigma}_2^2(\hat{\sigma}_1\hat{\sigma}_2-4\hat{\sigma}_3)\hat{\sigma}_3^2+(3\hat{\sigma}_1\hat{\sigma}_2^4-3\hat{\sigma}_2^2(\hat{\sigma}_1^2+2\hat{\sigma}_2)\hat{\sigma}_3 && (4.80)\\
&+ 4\hat{\sigma}_1(3\hat{\sigma}_1^2-7\hat{\sigma}_2)\hat{\sigma}_3^2)\hat{\sigma}_4-(\hat{\sigma}_1(4\hat{\sigma}_1^2-15\hat{\sigma}_2)\hat{\sigma}_2+12(\hat{\sigma}_1^2-2\hat{\sigma}_2)\hat{\sigma}_3)\hat{\sigma}_4^2] \\
&+ C_{[1,2]}[-(\hat{\sigma}_1^2-2\hat{\sigma}_2)^2\hat{\sigma}_3^3-\hat{\sigma}_1(\hat{\sigma}_1^2-\hat{\sigma}_2)\hat{\sigma}_2^3\hat{\sigma}_4+\hat{\sigma}_3(\hat{\sigma}_2(4\hat{\sigma}_1^4-5\hat{\sigma}_1^2\hat{\sigma}_2+3\hat{\sigma}_2^2) \\
&- \hat{\sigma}_1(9\hat{\sigma}_1^2-8\hat{\sigma}_2)\hat{\sigma}_3)\hat{\sigma}_4-\hat{\sigma}_4^2(\hat{\sigma}_1(4\hat{\sigma}_1^2-\hat{\sigma}_2)\hat{\sigma}_2-3(7\hat{\sigma}_1^2-4\hat{\sigma}_2)\hat{\sigma}_3+11\hat{\sigma}_1\hat{\sigma}_4)], \\
Q_{4;\mathrm{bulk}}^{\hat{\mathcal{O}}_2|2,2}(\{y\}_2,\{y'\}_2) &= \hat{\sigma}_2\sigma_1^3(K^{[0,2]}\sigma_1^2+\hat{\sigma}_2(\hat{\sigma}_2+\hat{\sigma}_1\sigma_1))+4\hat{\sigma}_2^2\sigma_1(\hat{\sigma}_2-\hat{\sigma}_1\sigma_1)\sigma_2 \\
&+ \hat{\sigma}_2\sigma_1(\hat{\sigma}_1^4-\hat{\sigma}_2\sigma_1^2)\sigma_2+3\hat{\sigma}_1\hat{\sigma}_2^2\sigma_2^2+3\hat{\sigma}_1\hat{\sigma}_2(\sigma_1^2-\sigma_2)\sigma_2(\hat{\sigma}_1^2+\sigma_2) \\
&+ K^{[0,2]}\hat{\sigma}_1\sigma_1^4(\hat{\sigma}_2+\sigma_2)-5\hat{\sigma}_2\sigma_1\sigma_2(\hat{\sigma}_1^2(\hat{\sigma}_2-\sigma_1^2)+\hat{\sigma}_2\sigma_2) \\
&- \sigma_1\sigma_2(\hat{\sigma}_1^4\sigma_1^2-\hat{\sigma}_2\sigma_2^2)+Z^{[2,2]}\hat{A}_{[3,0]}(K^{[2,0]}\hat{\sigma}_2(\hat{\sigma}_1+\sigma_1)-K^{[0,2]}\sigma_1^3) \\
&+ K^{[0,2]}Z^{[2,2]}((\hat{A}_{[3,0]}+C_{[0,3]}-1)\sigma_1^3+\hat{\sigma}_1(C_{[0,3]}\sigma_2-K^{[2,0]}B_{[2,2]})). && (4.81)
\end{aligned}
$$

Equations (4.74)-(4.81) provide a set of solutions to the bulk form factor equations hitherto unknown.

5 Conclusions and outlook

In this paper we have initiated the boundary form factor program for theories with many particles, concentrating on the example of A_n-ATFTs. We have computed all minimal one- and two-particle form factors and provided a description of the pole structure of higher particle form factors for these theories. We have then specialize our study to the self-dual point, $B=1$, and to the A_2-theory, which possesses a pair of particles, $1 = \bar{2}$, that can be regarded as bound states resulting from the processes $1 + 1 \to 2$ and $2 + 2 \to 1$. The presence of bound states implies that besides the kinematic residue equations also the bound state residue equations need to be satisfied, which makes the computation of form factors a lot more involved.

For the A_2-case we have obtained all form factors, up to four particles, of two families of fields which correspond to spinless and spin-1 fields of the bulk theory, respectively. Indeed we have shown that the form factors of our first family of solutions reduce to those obtained in [32] for bulk spinless fields in the appropriate limit. In addition, we have obtained all form factors up to four particles of another class of fields which correspond to spin-1 bulk fields other than simple derivatives of the previous ones. The bulk form factors of these fields were not known up to now and have been obtained as a certain limit of our boundary solutions.

This work provides further support to the statement that the boundary form factor program [15] is a very useful tool for the computation of form factors of boundary fields, even for multi-particle theories. The fact remains however that the structure of these solutions is a lot more involved than for the bulk case and identifying general patterns is still a challenge, even for single particle models.

We would like to finish by emphasizing that the boundary form factor program for IQFTs is still at its early stages of development. The study of more models should provide further insight into the mathematical structures of the form factors and applications of the program to the computation of correlation functions should be further explored. Focusing on the present work, a natural follow up would be to look at ATFTs related to other simple Lie algebras, including the completion of the study of the A_n-case which we hope to carry out in a near future.

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